Department of Computer Science and Engineering
(CS, IOT, DS, AIML)

Material

II B. Tech I Semester
Subject: Discrete Mathematics

Code: A0510

Academic Year 2021-22
Regulations: MR 20

<table>
<tr><td>2020-21
Onwards
(MR-20)</td><td colspan="1">MALLA REDDY ENGINEERING COLLEGE
(Autonomous)</td><td colspan="3">B.Tech.
III Semester</td></tr>
<tr><td>Code: A0507</td><td rowspan="2">Discrete Mathematics
(Common for CSE, CSE (Cyber Security), CSE (AI and ML), CSE(DS), CSE(IOT) and IT)</td><td>L</td><td>T</td><td>P</td></tr>
<tr><td>Credits: 3</td><td>3</td><td>-</td><td>-</td></tr>
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Prerequisites: NIL

Course Objectives:

This course provides the concepts of mathematical logic demonstrate predicate logic and Binary Relations among different variables, discuss different type of functions and concepts of Algebraic system and its properties. It also evaluates techniques of Combinatorics based on counting methods and analyzes the concepts of Generating functions to solve Recurrence equations.

MODULE I: Mathematical Logic [10 Periods]

Basic Logics - Statements and notations, Connectives, Well-formed formulas, Truth Tables, tautology.

Implications and Quantifiers - Equivalence implication, Normal forms, Quantifiers, Universal quantifiers.

MODULE II: Predicate Logic and Relations [10 Periods]

Predicate Logic - Free & Bound variables, Rules of inference, Consistency, proof of contradiction, Proof of automatic Theorem.

Relations - Properties of Binary Relations, equivalence, transitive closure, compatibility and partial ordering relations, Lattices, Hasse diagram.

MODULE III: Functions and Algebraic Structures [10 Periods]

A: Functions - Inverse Function, Composition of functions, recursive Functions - Lattice and its Properties.

B: Algebraic structures - Algebraic systems Examples and general properties, Semi-groups and monoids, groups, sub-groups, homomorphism, Isomorphism, Lattice as POSET, Boolean algebra.

MODULE IV: Counting Techniques and Theorems [09 Periods]

Counting Techniques - Basis of counting, Combinations and Permutations with repetitions, Constrained repetitions

Counting Theorems - Binomial Coefficients, Binomial and Multinomial theorems, principles of Inclusion – Exclusion. Pigeon hole principle and its applications.

MODULE V: Generating functions and Recurrence Relation [09 Periods]

Generating Functions - Generating Functions, Function of Sequences, Calculating Coefficient of generating function.

Recurrence Relations - Recurrence relations, Solving recurrence relation by substitution and Generating functions. Method of Characteristics roots, solution of Non-homogeneous Recurrence Relations.

TEXTBOOKS:

1. J P Tremblay & R Manohar, "**Discrete Mathematics with applications to Computer Science**", Tata McGraw Hill.
2. J.L. Mott, A. Kandel, T.P.Baker "**Discrete Mathematics for Computer Scientists & Mathematicians**", PHI.

REFERENCES:

1. Kenneth H. Rosen, "**Discrete Mathematics and its Applications**", TMH, Fifth Edition.
2. Thomas Koshy, "**Discrete Mathematics with Applications**", Elsevier.
3. Grass Man & Trembley, "**Logic and Discrete Mathematics**", Pearson Education.
4. C L Liu, D P Nohapatra, "**Elements of Discrete Mathematics - A Computer Oriented Approach**", Tata McGraw Hill, Third Edition.

E-RESOURCES:

1. http://www.cse.iitd.ernet.in/~bagchi/courses/discrete-book/fullbook.pdf
2. http://www.medellin.unal.edu.co/~curmat/matdiscretas/doc/Epp.pdf
3. http://ndl.iitkgp.ac.in/document/yVCWqd6u7wgye1qwH9xY7xPG734QA9tMJN2ncqS12ZbN7pUSSlWCxSgPOZJEokyWJlxQLYsrFyeITA70W9C8Pg
4. http://nptel.ac.in/courses/106106094/

Course Outcomes:

At the end of the course, a student will be able to

1. **Apply** the concepts of connectives and normal forms in real time applications.
2. **Summarize** predicate logic, relations and their operations.
3. **Describe** functions, algebraic systems, groups and Boolean algebra.
4. **Illustrate** practical applications of basic counting principles, permutations, combinations, and the pigeonhole methodology.
5. **Analyze** techniques of generating functions and recurrence relations.

CO- PO, PSO Mapping
(3/2/1 indicates strength of correlation) 3-Strong, 2-Medium, 1-Weak

COs	Programme Outcomes (POs)												PSOs		
	PO1	PO2	PO3	PO4	PO5	PO6	PO7	PO8	PO9	PO10	PO11	PO12	PSO1	PSO2	PSO3
CO1	2				3							2	3		
CO2	3											2	3		
CO3		3										2	3		
CO4	3	3	2	3								2		3	
CO5					3							2		3	

7 copies.

Generating functions:—

Consider a sequence of real numbers $a_0\, a_1\, a_2\, a_3 \ldots$ let us denote this sequence by $\langle a_r \rangle$. Given this sequence, suppose there exists a function $f(x)$ whose expansion in a series of powers of x is

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_{n-1} x^{n-1} + \ldots = \sum_{r=0}^{\infty} a_r x^r \quad \text{—— (1)}$$

Then $f(x)$ is called a generating function for the sequence $\langle a_r \rangle$.

In other words, given a sequence $\langle a_r \rangle$, if there exists a function $f(x)$ such that a_r is the coefficient of x^r in the expansion of $f(x)$ in a series of powers of x, then $f(x)$ is called a generating function of $\langle a_r \rangle$.

If $f(x)$ is a generating function of the sequence $\langle a_r \rangle$, we say that $f(x)$ generates the sequence $\langle a_r \rangle$. The series on the right hand side of expression (1) is known as the power series expansion of $f(x)$.

Eg:- (i) $(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots = \sum_{r=0}^{\infty} x^r$.

$f(x) = (1-x)^{-1}$ is a generating function for the sequence $1, 1, 1, 1, \ldots$

(ii) $(1+x)^{-1} = 1 - x + x^2 - x^3 + \ldots = \sum_{r=0}^{\infty} (-1)^r . x^r$.

$f(x) = (1+x)^{-1}$ is a generating function for the sequence $1, -1, 1, -1, \ldots$

(iii) For any real number n we have the binomial expansion

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1.2} x^2 + \frac{n(n-1)(n-2)}{1.2.3} x^3 + \ldots = \sum_{r=0}^{\infty} \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!} x^r.$$

For this we note that $f(x) = (1+x)^n$ is a generating function for the sequence

$$1, \frac{n}{1!}, \frac{n(n-1)}{2!}, \frac{n(n-1)(n-2)}{3!} \ldots$$

If n is a positive integer the expansion given by (4) terminates with the term containing x^n. In this case $(1+x)^n$ generates the sequence

$$\binom{n}{0} \binom{n}{1} \binom{n}{2}, \ldots \binom{n}{n}, 0, 0, 0 \ldots$$

When n is a real number (not necessarily a +ve integer) suppose we define $\binom{n}{r}$ by $\binom{n}{0} = 1$. and $\binom{n}{r} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}$ for $r \geq 1$.

$$(1+x)^n = \sum_{r=0}^{n} \binom{n}{r} x^r.$$

Find the sequences generated by the following functions.

(a) $(3+x)^3$ (b) $2x^2(1-x)^{-1}$ (c) $\frac{1}{1-x} + 2x^3$ (d) $(1+3x)^{-\frac{1}{3}}$ (e) $3x^3 + e^{2x}$.

Sol:- (a) $(3+x)^3 = 27 + x^3 + 9x^2 + 27x$.

$(3+x)^3 = 27 + 27x + 9x^2 + x^3$.

The sequence generated by $(3+x)^3$ is $27, 27, 9, 1, 0, 0, 0, \ldots$

(b) $2x^2(1-x)^{-1} = 2x^2(1+x+x^2+\cdots)$.

$= 2x^2 + 2x^3 + 2x^4 + \cdots$

$= 0 + 0.x + 2x^2 + 2x^3 + 2x^4 + \cdots$

The sequence generated by $2x^2(1-x)^{-1}$ is $0, 0, 2, 2, \ldots$

(c). $\frac{1}{1-x} + 2x^3 = (1-x)^{-1} + 2x^3$.

$= (1+x+x^2+x^3+\cdots) + 2x^3$.

$= 1 + x + x^2 + 3x^3 + x^4 + x^5 + \cdots$

The sequence generated by $\frac{1}{1-x} + 2x^3$ is $1, 1, 1, 3, 1, 1, \ldots$

(d) $(1+3x)^{-\frac{1}{3}} = 1 + \sum_{r=1}^{\infty} \frac{\left(-\frac{1}{3}\right)\left(-\frac{1}{3}-1\right)\left(-\frac{1}{3}-2\right)\cdots\left(-\frac{1}{3}-(r-1)\right)}{r!}(3x)^r$.

$= 1 + \sum_{r=1}^{\infty} \frac{\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)\left(-\frac{7}{3}\right)\cdots\left(\frac{-3r+2}{3}\right)}{r!} 3^r x^r$.

$= 1 + \sum_{r=1}^{\infty} \frac{(-1)(-4)(-7)\cdots(-3r+2)}{r!} x^r$.

The sequence generated by the function $(1+3x)^{-\frac{1}{3}}$ is $1, -1, \frac{(-1)(-4)}{2}, \frac{(-1)(-4)(-7)}{3!}, \ldots$

(e) $3x^3 + e^{2x} = 3x^3 + 1 + \frac{2x}{1!} + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \frac{(2x)^4}{4!} + \cdots$

$= \cancel{3x^3} + 1 + \frac{2}{1!}x + \frac{2^2}{2!}x^2 + \frac{2^3}{3!}x^3 + \frac{2^4}{4!}x^4 + \cdots$

$= 1 + \frac{2}{1!}x + \frac{2^2}{2!}x^2 + \left(3 + \frac{2^3}{3!}\right)x^3 + \frac{2^4}{4!}x^4 + \cdots$

The sequence generated by the function is $3x^3 + e^{2x}$ is

$1, \frac{2}{1!}, \frac{2^2}{2!}, 3 + \frac{2^3}{3!}, \frac{2^4}{4!}, \frac{2^5}{5!}, \cdots$

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If n is a positive integer, prove the following

(a) $\binom{-n}{r} = (-1)^r \binom{n+r-1}{r} = (-1)^r \binom{n+r-1}{n-1}$ (b) $\binom{2n}{n} = \sum_{r=0}^{n} \binom{n}{r}^2$

Sol:- (a) $$\binom{-n}{r} = \frac{(-n)(-n-1)(-n-2)\cdots(-n-r+1)}{r!} \quad \left[\because \binom{n}{r} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}\right]$$

$$= (-1)^r \frac{n(n+1)(n+2)\cdots(n+r-1)}{r!}$$

$$= (-1)^r \frac{1.2.3\cdots(n-1)n(n+1)(n+2)\cdots(n+r-1)}{1.2.3\cdots(n-1).\,r!}$$

$$\binom{-n}{r} = (-1)^r \frac{(n+r-1)!}{(n-1)!\,r!} = (-1)^r \binom{n+r-1}{r} = (-1)^r \binom{n+r-1}{n-1}$$

(b) We note that $\binom{2n}{n}$ is the co efficient of x^n in the expansion of $(1+x)^{2n}$.

$$(1+x)^{2n} = (1+x)^n (1+x)^n = \sum_{r=0}^{n} \binom{n}{r} x^r \sum_{s=0}^{n} \binom{n}{s} x^s$$

The co efficient of x^n in the R.H.S of this is.

$$= \left[\binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n\right]\left[\binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n\right]$$

$$= \binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \binom{n}{2}\binom{n}{n-2} + \cdots + \binom{n}{n}\binom{n}{0}$$

$$= \sum_{r=0}^{n} \binom{n}{r}\binom{n}{n-r} = \sum_{r=0}^{n} \binom{n}{r}\binom{n}{r} = \sum_{r=0}^{n} \binom{n}{r}^2$$

$$\therefore \binom{2n}{n} = \sum_{r=0}^{n} \binom{n}{r}^2.$$

If n is a positive integer, prove that $(1-x)^{-n} = \sum_{r=0}^{\infty} \binom{n+r-1}{r} x^r$.

Sol:- We have $\binom{n+r-1}{r} = (-1)^r \binom{-n}{r}$

$$\sum_{r=0}^{\infty} \binom{n+r-1}{r} x^r = \sum_{r=0}^{\infty} (-1)^r \binom{-n}{r} x^r = \sum_{r=0}^{\infty} \binom{-n}{r} (-x)^r.$$

$$= (1-x)^{-n}.$$

Let $f(x) = (1+x+x^2)(1+x)^n$ where n is a positive integer. Find the coefficient of x^7, x^8 and x^k for $0 \le k \le n+2$ in $f(x)$.

sol: Given that $f(x) = (1+x+x^2)(1+x)^n$

$$= (1+x+x^2)\sum_{r=0}^{n}\binom{n}{r}x^r$$

$$f(x) = \sum_{r=0}^{n}\binom{n}{r}x^r + \sum_{r=0}^{n}\binom{n}{r}x^{r+1} + \sum_{r=0}^{n}\binom{n}{r}x^{r+2}.$$

(i) The coefficient of x^7 in $f(x)$ is.

$$f(x) = \left[\binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \dots + \binom{n}{7}x^7 + \dots \binom{n}{n}x^n\right] + \left[\binom{n}{0}x + \binom{n}{1}x^2 + \dots + \binom{n}{6}x^7 + \binom{n}{n}x^{n+1}\right]$$
$$+\left[\binom{n}{0}x^2 + \binom{n}{1}x^3 + \dots + \binom{n}{5}x^7 + \dots + \binom{n}{n}x^{n+2}\right]$$

$\therefore$ The coefficient of x^7 in $f(x)$ is $\binom{n}{7} + \binom{n}{6} + \binom{n}{5}$.

(ii) The coefficient of x^8 in $f(x)$.

$$f(x) = \left[\binom{n}{0} + \binom{n}{1}x + \dots + \binom{n}{8}x^8 + \dots \binom{n}{n}x^n\right] + \left[\binom{n}{0}x + \dots + \binom{n}{7}x^8 + \dots + \binom{n}{n}x^{n+1}\right] + \left[\binom{n}{0}x^2 + \dots +\right.$$
$$\left.\binom{n}{6}x^8 + \dots + \binom{n}{n}x^{n+2}\right].$$

$\therefore$ The coefficient of x^8 in $f(x)$ is $\binom{n}{8} + \binom{n}{7} + \binom{n}{6}$.

(iii) The coefficient of x^k in $f(x)$.

$$f(x) = \left[\binom{n}{0} + \binom{n}{1}x + \dots + \binom{n}{k}x^k + \dots + \binom{n}{n}x^n\right] + \left[\binom{n}{0}x + \dots + \binom{n}{k-1}x^k + \dots + \binom{n}{n}x^n\right] +$$
$$\left[\binom{n}{0}x^2 + \dots + \binom{n}{k-2}x^k + \dots + \binom{n}{n}x^{n+2}\right]$$

$\therefore$ The coefficient of x^k in $f(x)$ is $\binom{n}{k} + \binom{n}{k-1} + \binom{n}{k-2}$.

Determine the co efficient of (a) x^{12} in $x^3(1-2x)^{10}$.

(b) x^0 in $\left(3x^2-\left(\frac{2}{x}\right)\right)^{15}$ (c) x^5 in $(1-2x)^{-7}$ (d) x^{10} in $(x^3-5x)/(1-x)^3$.

(e) x^{15} in $(1+x)^4/(1-x)^4$ (f) x^8 in $1/(x-3)(x-2)^2$.

(g) x^{20} in $(x^2+x^3+x^4+x^5+x^6)^5$.

Sol:- (a) $x^3(1-2x)^{10} = x^3\sum_{r=0}^{10}\binom{10}{r}(-2x)^r = \sum_{r=0}^{10}\binom{10}{r}(-2)^r x^{r+3}$.

$$x^3(1-2x)^{10} = \left[\binom{10}{0}x^3+\binom{10}{1}(-2)x^4+\cdots+\binom{10}{9}(-2)^9x^{12}+\binom{10}{10}(-2)^{10}x^{13}\right]$$

The co efficient of x^{12} in $x^3(1-2x)^{10}$ is $-2^9\binom{10}{9} = -5120$.

(b) $\left(3x^2-\left(\frac{2}{x}\right)\right)^{15} = \sum_{r=0}^{15}\binom{15}{r}(3x^2)^{15-r}\left(-\frac{2}{x}\right)^r = \sum_{r=0}^{15}\binom{15}{r}(-2)^r 3^{15-r}(x^2)^{15-r}x^{-r}$

$$= \sum_{r=0}^{15}(-2)^r 3^{15-r}x^{30-3r}\binom{15}{r}.$$

$$= \binom{15}{0}3^{15}x^{30}+\binom{15}{1}(-2)3^{14}x^{27}+\cdots+\binom{15}{10}(-2)^{10}3^5+\cdots+\binom{15}{15}(-2)^{15}x^{-15}$$

$\therefore$ The co efficient of x^0 in $\left(3x^2-\left(\frac{2}{x}\right)\right)^{15}$ is $3^5.2^{10}\binom{15}{10}$.

(c) We know that If n is a positive integer,

$$(1-x)^{-n} = \sum_{r=0}^{\infty}\binom{n+r-1}{r}x^r.$$

$$(1-2x)^{-7} = \sum_{r=0}^{\infty}\binom{7+r-1}{r}(2x)^r = \sum_{r=0}^{\infty}2^r\binom{7+r-1}{r}x^r.$$

$$= \binom{6}{0}+2\binom{7}{1}x+2^2\binom{8}{2}x^2+\cdots+2^5\binom{11}{5}x^5+\cdots$$

$\therefore$ The coefficient of x^5 in $(1-2x)^{-7}$ is $2^5\binom{11}{5} = 2^5.\frac{11!}{5!.6!}$

(d) We know that If n is a positive integer, $(1-x)^{-n} = \sum_{r=0}^{\infty} \binom{n+r-1}{r} x^r$

$$\frac{x^3-5x}{(1-x)^3} = (x^3-5x)(1-x)^{-3} = (x^3-5x)\sum_{r=0}^{\infty}\binom{3+r-1}{r}x^r = (x^3-5x)\sum_{r=0}^{\infty}\binom{2+r}{r}x^r$$

$$= \sum_{r=0}^{\infty}\binom{2+r}{r}x^{r+3} - 5\sum_{r=0}^{\infty}\binom{2+r}{r}x^{r+1}.$$

$$= \left[\binom{2}{0}x^3 + \binom{3}{1}x^4 + \binom{4}{2}x^5 + \cdots + \binom{9}{7}x^{10} + \cdots\right] - 5\left[\binom{2}{0}x + \binom{3}{1}x^2 + \cdots + \binom{11}{9}x^{10} + \cdots\right]$$

$\therefore$ The coefficient of x^{10} in $\frac{x^3-5x}{(1-x)^3}$ is $\binom{9}{7} - 5\binom{11}{9} = \frac{9!}{7!\,2!} - 5\,\frac{11!}{9!\,2!} =$

(e) We know that If n is a positive integer $(1-x)^{-n} = \sum_{r=0}^{\infty}\binom{n+r-1}{r}x^r$.

$$\frac{(1+x)^4}{(1-x)^4} = (1+x)^4(1-x)^{-4} = \sum_{r=0}^{4}\binom{4}{r}x^r \times (1-x)^{-4}$$

$$= (1+4x+6x^2+4x^3+x^4)(1-x)^{-4}.$$

$$= (1+4x+6x^2+4x^3+x^4)\sum_{r=0}^{\infty}\binom{4+r-1}{r}x^r$$

$$= (1+4x+6x^2+4x^3+x^4)\sum_{r=0}^{\infty}\binom{3+r}{r}x^r.$$

$$= \sum_{r=0}^{\infty}\binom{3+r}{r}x^r + 4\sum_{r=0}^{\infty}\binom{3+r}{r}x^{r+1} + 6\sum_{r=0}^{\infty}\binom{3+r}{r}x^{r+2} + 4\sum_{r=0}^{\infty}\binom{3+r}{r}x^{r+3}.$$
$$+ \sum_{r=0}^{\infty}\binom{3+r}{r}x^{r+4}.$$

$\therefore$ The coefficient of x^{15} in $\frac{(1+x)^4}{(1-x)^4}$ is $\binom{18}{15} + 4\binom{17}{14} + 6\binom{16}{13} + 4\binom{15}{12} + \binom{14}{11}$

(f) We know that If n is a positive integer $(1-x)^{-n} = \sum_{r=0}^{\infty}\binom{n+r-1}{r}x^r$

$$\frac{1}{(x-3)(x-2)^2} = \frac{1}{(-3)\left(1-\frac{x}{3}\right)(-2)^2\left(1-\frac{x}{2}\right)^2} = \frac{-1}{12}\left(1-\frac{x}{3}\right)^{-1}\left(1-\frac{x}{2}\right)^{-2}.$$

$$= \frac{-1}{12}\sum_{r=0}^{\infty}\left(\frac{x}{3}\right)^r \cdot \sum_{s=0}^{\infty}\binom{1+s}{s}\left(\frac{x}{2}\right)^s.$$

$$= \frac{-1}{12}\sum_{r=0}^{\infty}\left(\frac{1}{3}\right)^r x^r \sum_{s=0}^{\infty}\left(\frac{1}{2}\right)^s \binom{1+s}{s}x^s.$$

$$= -\frac{1}{12}\left\{\left[\left(\frac{1}{3}\right)^0 + \left(\frac{1}{3}\right)x + \left(\frac{1}{3}\right)^2 x^2 + \cdots\right]\cdot\left[\binom{1}{0}\left(\frac{1}{2}\right)^0 + \left(\frac{1}{2}\right)\binom{2}{0}x + \binom{3}{2}x^2\left(\frac{1}{2}\right)^2 + \cdots\right]\right\}.$$

The Co efficient of x^8 in $\frac{1}{(x-3)(x-2)^2}$ is

$$-\frac{1}{12}\left[\left(\frac{1}{3}\right)^0\left(\frac{1}{2}\right)^8\binom{9}{8} + \left(\frac{1}{3}\right)^1\left(\frac{1}{2}\right)^7\binom{8}{7} + \left(\frac{1}{3}\right)^2\left(\frac{1}{2}\right)^6\binom{7}{6} + \cdots + \left(\frac{1}{3}\right)^7\left(\frac{1}{2}\right)\binom{2}{1} + \left(\frac{1}{3}\right)^8\left(\frac{1}{2}\right)^0\binom{1}{0}\right]$$

$$= -\frac{1}{12}\sum_{k=0}^{8}\left(\frac{1}{3}\right)^k\left(\frac{1}{2}\right)^{8-k}\binom{9-k}{8-k}.$$

(g) We have $(x^2+x^3+x^4+x^5+x^6)^5 = \left[x^2(1+x+x^2+x^3+x^4)\right]^5 =$

$$= x^{10}(1+x+x^2+x^3+x^4)^5$$

$$= x^{10}\frac{(1-x)^5(1+x+x^2+x^3+x^4)^5}{(1-x)^5}$$

$$= x^{10}\frac{\left[(1-x)(1+x+x^2+x^3+x^4)\right]^5}{(1-x)^5} = \frac{x^{10}(1-x^5)^5}{(1-x)^5}.$$

$$= x^{10}.(1-x^5)^5(1-x)^{-5}$$

[We know that If n is a +ve Integer $(1-x)^{-n} = \sum_{r=0}^{\infty}\binom{n+r-1}{r}x^r$]

$$= x^{10}\sum_{r=0}^{5}\binom{5}{r}(-x^5)^r\sum_{s=0}^{\infty}\binom{5+s-1}{s}x^s.$$

$$= x^{10}\sum_{r=0}^{5}\binom{5}{r}(-x^5)^r\sum_{s=0}^{\infty}\binom{4+s}{s}x^s.$$

$$= x^{10}.\left[\binom{5}{0} + \binom{5}{1}(-x^5) + \binom{5}{2}x^{10} + \cdots + \binom{5}{5}(-x^{25})\right]\left[\binom{4}{0} + \binom{5}{1}x + \binom{5}{2}x^2 + \binom{5}{3}x^3 + \cdots + \binom{14}{10}x^{10} + \cdots\right].$$

The co efficient of x^{20} in $(x^2+x^3+x^4+x^5+x^6)^5$ is

$$\binom{5}{0}\binom{14}{10} - \binom{5}{1}\binom{9}{5} + \binom{5}{2}\binom{4}{0} = \binom{14}{10} - 5\binom{9}{5} + \binom{5}{2}.$$

Find the co efficient of x^{27} in the following functions.

(a) $(x^4+x^5+x^6+\cdots)^5$ (b) $(x^4+2x^5+3x^6+\cdots)^5$.

sol:- (a) We have $(x^4+x^5+x^6+\cdots)^5 = [x^4(1+x+x^2+\cdots)]^5$

$$= x^{20}(1+x+x^2+\cdots)^5$$

$$= x^{20}[(1-x)^{-1}]^5 = x^{20}(1-x)^{-5}$$

[$\because$ We know that If n is +ve integer $(1-x)^{-n} = \sum_{r=0}^{\infty}\binom{n+r-1}{r}x^r$]

$$= x^{20}\sum_{r=0}^{\infty}\binom{5+r-1}{r}x^r$$

$$= x^{20}\sum_{r=0}^{\infty}\binom{r+4}{r}x^r$$

$$= \sum_{r=0}^{\infty}\binom{4+r}{r}x^{20+r}.$$

The co efficient of x^{27} in $(x^4+x^5+\cdots)^5$ is $\binom{11}{7} = \frac{11!}{7!\,4!} = 330$.

(b) We have $(x^4+2x^5+3x^6+\cdots)^5 = [x^4(1+2x+3x^2+\cdots)]^5$

$$= x^{20}(1+2x+3x^2+\cdots)^5$$

$$= x^{20}[(1-x)^{-2}]^5$$

$$= x^{20}(1-x)^{-10}.$$

[$\because$ We know that If n is +ve integer $(1-x)^{-n} = \sum_{r=0}^{\infty}\binom{n+r-1}{r}x^r$]

$$= x^{20}\sum_{r=0}^{\infty}\binom{10+r-1}{r}x^r$$

$$= x^{20}\sum_{r=0}^{\infty}\binom{r+9}{r}x^r$$

$$= \sum_{r=0}^{\infty}\binom{9+r}{r}x^{20+r}.$$

The co efficient of x^{27} in $(x^4+2x^5+\cdots)^5$ is $\binom{16}{7} = \frac{16!}{9!\,7!} = 11,440$.

Find the co efficient of x^n in the following functions.

(a) $(1+x^2+x^4+\cdots)^7$. (b) $(x^2+x^3+x^4+\cdots)^4$ (c) $(x^8+x^9+x^{10}+\cdots)^9$.

Sol:- (a) We have $(1+x^2+x^4+\cdots)^7 = ((1-x^2)^{-1})^7 = (1-x^2)^{-7}$.

$\because$ We know that If n is +ve integer, $(1-x)^{-n} = \sum_{r=0}^{\infty} \binom{n+r-1}{r} x^r$.

$$= (1-x^2)^{-7} = \sum_{r=0}^{\infty} \binom{7+r-1}{7} (x^2)^r$$

$$= \sum_{r=0}^{\infty} \binom{6+r}{r} (x^2)^r.$$

From this we obesrve that when n is odd, the co efficient of x^n is zero and when n is even say 2m, the co efficient is. $\binom{6+m}{m}$

$$a_3 = ca_2 + f(3) = c\{c^2a_0 + cf(1) + f(2)\} + f(3).$$

$$a_3 = c^3a_0 + c^2f(1) + cf(2) + f(3) \quad \text{and so on}.$$

We obtain, by induction.

$$a_n = c^n a_0 + c^{n-1}f(1) + c^{n-2}f(2) + \cdots + c^{n-(n-1)}f(n-1) + c^{n-n}f(n).$$

$$a_n = c^n a_0 + c^{n-1}f(1) + c^{n-2}f(2) + \cdots + cf(n-1) + f(n).$$

$$a_n = c^n a_0 + \sum_{k=1}^{n} c^{n-k}f(k) \quad \text{for } n \geq 1 \text{ --- } (3).$$

This is the general solution of the recurrence relation (2) which is equivalent to the relation (1).

If $f(n) = 0$ that is if the recurrence relation is homogeneous, the solution (3) becomes. $a_n = c^n a_0$ for $n \geq 1$ --- (4).

The solutions (3) and (4) yield particular solutions if a_0 is specified. The specified value of a_0 is called the initial condition.

1) Solve the recurrence relation $a_{n+1} = 4a_n$ for $n \geq 0$, given that $a_0 = 3$.

Sol:- We consider for solution linear recurrence relations of the form

$$a_n = ca_{n-1} + f(n), \quad \text{for } n \geq 1. \text{ --- } (1).$$

Where c is a constant and $f(n)$ is a known function which is a first order recurrence relation with constant co efficients.

If $f(n) = 0$, the relation is called homogeneous otherwise it is called non homogeneous.

The relation (1) can be solved in a trivial way.

We note that this relation may be re written as (by changing n to $n+1$)

$$a_{n+1} = ca_n + f(n+1), \quad \text{for } n \geq 0 \text{ --- } (2).$$

For $n = 0, 1, 2, 3, \ldots$

$$a_1 = ca_0 + f(1)$$

$$a_2 = ca_1 + f(2) = c\{ca_0 + f(1)\} + f(2)$$

$$a_2 = c^2 a_0 + cf(1) + f(2).$$

$$a_3 = ca_2 + f(3) = c^3 a_0 + c^2 f(1) + cf(2) + f(3) \text{ and so on}$$

We obtain by induction.

$$a_n = c^n a_0 + c^{n-1} f(1) + c^{n-2} f(2) + \cdots + c^{n-(n-1)} f(n-1) + c^{n-n} f(n).$$

$$a_n = c^n a_0 + c^{n-1} f(1) + c^{n-2} f(2) + \cdots + cf(n-1) + f(n).$$

$$a_n = c^n a_0 + \sum_{k=1}^{n} c^{n-k} f(k) \text{ for } n \geq 1 \text{ ——— } (3).$$

This is the general solution of the recurrence relation (2). which is equi-valent to the relation (1).

If $f(n) = 0$, that is if the recurrence relation is homo. the solution (3) becomes. $a_n = c^n a_0$ for $n \geq 1$. ——— (4).

Given that the recurrence relation. $a_{n+1} = 4^n a_n$ for $n \geq 1$.

Which is linear, first order and homogeneous recurrence relation.

The general solution of the linear, first order homo. recurrence relation is $a_n = 4^n a_0$ for $n \geq 1$ ——— (5). ($\because$ from (4))

Given that $a_0 = 3$.

Sub. $a_0 = 3$ in (5), we get

$$a_n = 3 \cdot 4^n \text{ for } n \geq 1.$$

This is the particular solution of the given relation satisfying the initial condition $a_0 = 3$.

Recurrence Relations :—

A recurrence relation is a formula that relates for any integer $n \geq 1$, the nth term of a sequence $A = \{a_r\}_{r=0}^{\infty}$ to one or more of the terms $a_0, a_1, a_2 \ldots a_{n-1}$. (OR). A sequence $\langle a_r \rangle$ may be defined by indicating a relation connecting its general term a_n with a_{n-1}, a_{n-2}, a_{n-3} etc. Such a relation is called recurrence relation for the sequence.

One can carry out a step by step computation to determine a_n from a_{n-1}, $a_{n-2}, a_{n-3} \ldots$ provided that the values of the function at one or more points are given. The given values are called initial conditions or boundary conditions of the recurrence relation.

→ The process of determining a_n from a recurrence relation is called solving of the relation. A value a_n that satisfies a recurrence relation is called its general ~~g~~ solution.

→ If the values of some particular terms of the sequence are specified. then by making use of these values in the general solution we obtain the particular solution that uniquely determines the sequence.

Eg:-(i) The numeric function (5, 8, 11, 14, ...) is defined by the recurrence relation $a_n = a_{n-1} + 3$, $n \geq 1$. with initial condition $a_0 = 5$.

(ii) The recurrence relation of the Fibonacci sequence of number (1, 1, 2, 3, 5, 8, 13, ...) is defined by $f_n = f_{n-1} + f_{n-2}$, $n \geq 2$ with initial Condition $f_0 = 1$. $f_1 = 1$.

The order of a recurrence relation :—

The order of recurrence relation is the difference between the largest and smallest subscript appearing in the relation.

Eg:- (i) $a_n = -3a_{n-1}$ is a recurrence relation of order 1.

(ii) $a_{n+2} - a_{n+1} - 2a_n = 0$ is a recurrence relation of order 2.

(iii) $a_n + a_{n-1} - 6a_{n-2} = 0$ is a recurrence relation of order 2.

Linear Recurrence Relation with Constant co efficients :—

A recurrence relation of the form. $c_0 a_n + c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} = f(n)$. Where c_i's are constants is called a linear recurrence relation of k^{th} order, provided that both c_0 and c_k are non zero.

Linear refers to the fact that every term containing an a_i has exactly one such factor and it occurs to the first power. The words Constant co-efficients mean each of the c_i's is a constant.

→ If $f(n)$ is identically zero, the relation is known as homogeneous

→ A recurrence relation is said to be linear non homogeneous if $f(n) \neq 0$.

Eg:- (a) The recurrence relation $a_n = 2a_{n-1}$ is a linear homogeneous relation with constant co efficient of degree 1.

(b) The recurrence relation $a_n = 2a_{n-1} a_{n-2}$ is not a linear homogeneous relation with constant Co efficient as term such as $a_{n-1}a_{n-2}$ is not permi-tted. Each term is to be of the form can.

(c) The recurrence relation $a_n - a_{n-1} = 6$ is not a linear homo. relation with constant co efficient because the expression of the right hand side is not zero.

(d) The recurrence relation $a_n + a_{n+1} + a_{n+2} = 0$ is a linear homoge-neous relation with constant co efficient of order 2.

Solution of Linear Recurrence Relation :—

Suppose we have a sequence that satisfies a certain recurrence relation and initial conditions. An explicit formula which satisfy the recurrence. relation with initial condition is called a solution to the recurrence relation.

The three methods of solving recurrence relations are

(i) Iteration

(ii) characteristic roots.

(iii) Generating functions.

Iteration Method :-

In this method the recurrence relation for a_n is used repeatedly to solve for a general expression for a_n in terms of n.

First order Linear Recurrence Relation :—

We consider for solution recurrence relations of the form.

$$a_n = ca_{n-1} + f(n) \text{ for } n \geq 1 \quad \text{—— (1)}$$

Where c is known constant and $f(n)$ is a known function. Such a relation is called a linear recurrence relation. of first order with constant coefficient. If $f(n) = 0$, the relation is called homogeneous, otherwise it is called non homogeneous (or inhomogeneous)

The relation (1) can be solved in a trivial way.

We note that this relation may be rewritten as (by changing n to $n+1$)

$$a_{n+1} = ca_n + f(n+1). \text{ for } n \geq 0 \quad \text{—— (2)}$$

For $n = 0, 1, 2, 3, \ldots$

$$a_1 = ca_0 + f(1)$$

$$a_2 = ca_1 + f(2) = c[ca_0 + f(1)] + f(2)$$

$$a_2 = c^2a_0 + cf(1) + f(2).$$

Solve the recurrence relation. $a_n = n\, a_{n-1}$ for $n \geq 1$, given that $a_0 = 1$. 9

Sol:- Given that the recurrence relation $a_n = n\, a_{n-1}$ for $n \geq 1$. —— ①.

For $n = 1, 2, 3, \ldots$

$$a_1 = a_0$$
$$a_2 = 2a_1 = (2\times1)a_0.$$
$$a_3 = 3a_2 = (3\times2\times1)a_0.$$
$$a_4 = 4a_3 = (4\times3\times2\times1)a_0 \quad \text{and so on}$$

$\therefore$ The general solution given recurrence relation is

$$a_n = n!\, a_0 \text{ for } n \geq 1. \quad —— ②$$

Given that $a_0 = 1$,

Sub. $a_0 = 1$ in ②, we get $a_n = n!$ is the required solution.

(3) Solve the recurrence relation $a_n - 3a_{n-1} = 5\cdot 3^n$, for $n \geq 1$, given that $a_0 = 2$.

Sol:- We consider for solution recurrence relations of the form.

$$a_n = c\,a_{n-1} + f(n), \text{ for } n \geq 1 \quad —— ①.$$

Where c is a known constant and $f(n)$ is a known function. Such a relation is called a linear recurrence relation of 1st order with constant coefficient. If $f(n) = 0$, the relation is called homogeneous otherwise, it is called non homogeneous.

The relation ① can be solved in a trivial way.

This relation may be rewritten as (by changing n to $n+1$)

$$a_{n+1} = c\,a_n + f(n+1), \text{ for } n \geq 0. \quad —— ②.$$

For $n = 0, 1, 2, 3, \ldots$

$$a_1 = c\,a_0 + f(1)$$
$$a_2 = c\,a_1 + f(2) = c^2 a_0 + c f(1) + f(2).$$
$$a_3 = c\,a_2 + f(3) = c^3 a_0 + c^2 f(1) + c f(2) + f(3)$$

and so on

We obtain by induction.

$$a_n = c^n a_0 + c^{n-1} f(1) + c^{n-2} f(2) + \cdots + c^{n-(n-1)} f(n-1) + c^{n-n} f(n).$$

$$= c^n a_0 + c^{n-1} f(1) + \cdots + c f(n-1) + f(n)$$

$$a_n = c^n a_0 + \sum_{k=1}^{n} c^{n-k} f(k) \quad \text{for } n \geq 1. \quad \text{——} (3)$$

This is the general solution of the recurrence relation (2). Which is equivalent to the relation (1).

Given that the recurrence relation $a_n = 3a_{n-1} + 5 3^n$ for $n \geq 1$.

The given relation may be rewritten as (by changing n to n+1)

$$a_{n+1} = 3a_n + 5\, 3^{n+1}. \quad \text{for } n \geq 0.$$

$$a_{n+1} = 3a_n + 5 f(n+1). \quad \text{Where } f(n) = 5 \cdot 3^n.$$

The general solution of this non homogeneous relation is.

$$a_n = c^n a_0 + \sum_{k=1}^{n} c^{n-k} f(k).$$

$$a_n = 3^n a_0 + \sum_{k=1}^{n} 3^{n-k} f(k).$$

The given initial condition $a_0 = 2$.

$$a_n = 3^n 2 + \sum_{k=1}^{n} 3^{n-k} f(k).$$

~~Substituting for f(n), n=1,2,3, ... this becomes.~~

$$a_n = 2 \cdot 3^n + 3^{n-1} f(1) + 3^{n-2} f(2) + 3^{n-3} f(3) + \cdots + 3^0 f(n).$$

Substituting for $f(n)$, $n = 1, 2, 3 \ldots$ this becomes

$$a_n = 2 \cdot 3^n + 3^{n-1} 5 \cdot 3 + 3^{n-2} 5 \cdot 3^2 + 3^{n-3} 5 \cdot 3^3 + \cdots + 5 \cdot 3^n.$$

$$a_n = 2 \cdot 3^n + 5\{3^n + 3^n + \cdots + 3^n\}.$$

$$a_n = 2 3^n + 5n 3^n = 3^n(5n+2).$$

This is the required solution of the given recurrence relation.

Solve the recurrence relation $a_n - a_{n-1} = 3n^2$ where $n \geq 1$ and $a_0 = 7$.

Sol:- Given that the recurrence relation $a_n = a_{n-1} + 3n^2$ for $n \geq 1$ —①.

The given relation may be re written as (changing n to n+1)

$$a_{n+1} = a_n + 3(n+1)^2 \quad \text{for } n \geq 0$$

$$a_{n+1} = a_n + 3f(n+1) \quad \text{where } f(n) = 3n^2.$$

The general solution of this non homogeneous relation is

$$a_n = c^n a_0 + \sum_{k=1}^{n} c^{n-k} f(k).$$

The given relation ① is of the form $a_n = ca_{n-1} + f(n)$

Here $c = 1$ $f(n) = 3n^2$.

$$a_n = 1^n a_0 + \sum_{k=1}^{n} 1^{n-k} \cdot f(k).$$

$$a_n = a_0 + \sum_{k=1}^{n} 3k^2$$

$$a_n = a_0 + [3 + 3.2^2 + 3.3^2 + 3.4^2 + \cdots + 3n^2]$$

$$a_n = a_0 + 3[1 + 2^2 + 3^2 + 4^2 + \cdots + n^2]$$

$$a_n = a_0 + 3 \cdot \frac{n(n+1)(2n+1)}{6}$$

$$a_n = a_0 + \frac{n(n+1)(2n+1)}{2}$$

$$a_n = 7 + \frac{n(n+1)(2n+1)}{2}$$

This is the required solution of the given recurrence relation.

Sol:- Solve the recurrence relation $a_{n+1} = a_n + (2n+3)$, $a_0 = 1$ $n \geq 0$.

Sol:- Given that the recurrence relation $a_{n+1} = a_n + (2n+3)$

$$a_{n+1} = a_n + 2(n+1) + 1, \quad n \geq 0.$$

Which is of the form $a_{n+1} = ca_n + f(n+1)$

Here $c = 1$, $f(n) = 2n+1$.

The general solution of this non homogeneous relation is

$$a_n = c^n a_0 + \sum_{k=1}^{n} c^{n-k} f(k)$$

$$a_n = 1^n a_0 + \sum_{k=1}^{n} 1^{n-k}.(2k+1)$$

$$a_n = a_0 + \sum_{k=1}^{n} (2k+1)$$

$$a_n = a_0 + \left[2\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1 \right]$$

$$= a_0 + 2\left[1+2+3+\cdots+n\right] + \left[1+1+1+\cdots 1\right]$$

$$= a_0 + 2.\frac{n(n+1)}{2} + n.$$

$$a_n = a_0 + n(n+1)+n$$

$$a_n = 1 + n^2+n+n \quad = 1+n^2+2n$$

$$a_n = 1+n(n+2).$$

Which is required solution of the given recurrence relation.

Solve the recurrence relation $a_n = a_{n-1} + \frac{n(n+1)}{2}$, $n \geq 1$.

Sol: Given that $a_n = a_{n-1} + \frac{n(n+1)}{2}$, $n \geq 1$ —— ①.

The given relation is of the form $a_n = c a_{n-1} + f(n)$, $n \geq 1$

Here $c=1$ $f(n) = \frac{n(n+1)}{2}$.

The given relation may be rewritten as (by changing n to n+1)

$$a_{n+1} = c a_n + f(n+1), \text{ for } n \geq 0.$$

$$a_{n+1} = a_n + \frac{(n+1)(n+2)}{2} \text{ for } n \geq 0.$$

The general solution of this non homogeneous relation is.

$$a_n = c^n a_0 + \sum_{k=1}^{n} c^{n-k} f(k).$$

$$a_n = a_0 + \sum_{k=1}^{n} 1.\frac{k(k+1)}{2}$$

$$a_n = a_0 + \sum_{k=1}^{n} \frac{(k^2+k)}{2}$$

$$a_n = a_0 + \frac{1}{2}\left[\sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} k\right]$$

$$= a_0 + \frac{1}{2}\left\{\left[1^2+2^2+3^2+\cdots+n^2\right] + \left[1+2+3+\cdots n\right]\right\}$$

$$= a_0 + \frac{1}{2}\left[\left[\frac{n(n+1)(2n+1)}{6}\right] + \left[\frac{n(n+1)}{2}\right]\right]$$

$$= a_0 + \frac{n(n+1)(2n+1)}{12} + \frac{n(n+1)}{4}$$

Which is the required solution of the given recurrence relation.

Solve the relation $a_n = a_{n-1} + n^3$, $a_0 = 5$.

Sol:- Given that $a_n = a_{n-1} + n^3$.

The given relation is of the form $a_n = c\,a_{n-1} + f(n)$

Here $c = 1$ $f(n) = n^3$

The given relation may be rewritten as (by changing n to $n+1$)

$$a_{n+1} = a_n + (n+1)^3$$

The general solution of this non homogeneous relation is.

$$a_n = c^n a_0 + \sum_{k=1}^{n} c^{n-k} f(k)$$

$$a_n = a_0 + \sum_{k=1}^{n} k^3$$

$$a_n = a_0 + \left[1^3+2^3+3^3+4^3+\cdots+n^3\right].$$

$$a_n = a_0 + \frac{n^2(n+1)^2}{4}.$$

$$a_n = 5 + \frac{n^2(n+1)^2}{4}. \quad (\because a_0 = 5)$$

Which is the required solution of the given relation.

Solve the recurrence relation $2a_{n+1} - a_n = 2$

Sol: Given that $2a_{n+1} - a_n = 2$ i.e. $a_{n+1} = \frac{1}{2}a_n + 1$.

It can be re written as (by changing n to $n-1$)

$$a_n = \frac{1}{2}a_{n-1} + 1$$

Which is of the form $a_n = c\,a_{n-1} + f(n)$

$$c = \frac{1}{2} \quad f(n) = 1.$$

The general solution of this non homogeneous relation is

$$a_n = c^n a_0 + \sum_{k=1}^{n} c^{n-k} f(k)$$

$$a_n = \left(\tfrac{1}{2}\right)^n a_0 + \sum_{k=1}^{n} \left(\tfrac{1}{2}\right)^{n-k} \cdot 1$$

$$= \left(\tfrac{1}{2}\right)^n a_0 + \left[\left(\tfrac{1}{2}\right)^{n-1} + \left(\tfrac{1}{2}\right)^{n-2} + \cdots + \left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{2}\right) + 1\right]$$

$$= \left(\tfrac{1}{2}\right)^n a_0 + \left[1 + \left(\tfrac{1}{2}\right) + \left(\tfrac{1}{2}\right)^2 + \cdots + \left(\tfrac{1}{2}\right)^{n-1}\right]$$

$$= \left(\tfrac{1}{2}\right)^n a_0 + \frac{1\left(1 - \left(\tfrac{1}{2}\right)^n\right)}{\left(1 - \tfrac{1}{2}\right)} \qquad \left[\because a_n = \frac{a_0(1-r^n)}{1-r}\right.$$

$$= \left(\tfrac{1}{2}\right)^n a_0 + 2\left(1 - \frac{1}{2^n}\right)$$

$$a_n = \left(\tfrac{1}{2}\right)^n a_0 + \left(2 - \frac{1}{2^{n-1}}\right)$$

Which is the required solution of the given recurrence relation.

Solve the recurrence relation $2a_n - 3a_{n-1} = 0,\ n \geq 1,\ a_4 = 81$.

Sol: Given that $2a_n - 3a_{n-1} = 0$ i.e $a_n = \frac{3}{2}a_{n-1}$

which is of the form $a_n = c\,a_{n-1} + f(n)$

$$c = \frac{3}{2} \quad f(n) = 0.$$

The given relation homogeneous recurrence relation.

The given relation may be re written as (by changing n to $n+1$)

$$a_{n+1} = \frac{3}{2} a_n + 0.$$

The general solution given recurrence relation is

$$a_n = c^n a_0 + \sum_{k=1}^{n} c^{n-k} f(k)$$

$$a_n = \left(\frac{3}{2}\right)^n . a_0 + 0 \qquad [\because f(n) = 0]$$

$$a_n = \left(\frac{3}{2}\right)^n a_0.$$

Given that $a_4 = 81$.

$$a_n = \left(\frac{3}{2}\right)^n a_0.$$

$$n=4, \quad a_4 = \left(\frac{3}{2}\right)^4 a_0.$$

$$81 = \left(\frac{3}{2}\right)^4 a_0$$

$$a_0 = 16.$$

$$a_n = \left(\frac{3}{2}\right)^n 16$$

$$\therefore a_n = \frac{3^n}{2^n} . 2^4$$

$$\therefore \boxed{a_n = \frac{3^n}{2^{n-4}}}$$

Find a_{12}, if $a_{n+1}^2 = 5a_n^2$ where $a_n > 0$ for $n \geq 0$ and $a_0 = 2$.

Sol:- Given that $a_{n+1}^2 = 5a_n^2$ — (1)

which is not linear in a_n.

To make it linear assume $b_n = a_n^2$.

Then the original recurrence relation becomes.

$$b_{n+1} = 5b_n, \; n \geq 0. \text{ — (2)}$$

which homogeneous recurrence relation.

which is of the form $a_{n+1} = c\, a_n$, Here $c = 5$

We know that the general solution of the given recurrence relation is

$$a_n = c^n a_0 + \sum_{k=1}^{n} c^{n-k} f(k).$$

If the given relation is homogeneous then its general solution is

$$a_n = c^n a_0 .$$

general solution of the relation (2) is

$$b_n = 5^n b_0 \quad \text{——— (3)}.$$

we have $b_n = a_n^2$

$b_0 = a_0^2 \Rightarrow b_0 = 4 \quad (\because a_0 = 2)$

From equation (1), we have $b_n = 4.5^n$

$$a_n^2 = 4.5^n$$

$n=12$, $\quad a_{12}^2 = 4.5^{12}$

$$a_{12} = 2(5^{12})^{1/2}$$

$$a_{12} = 2.5^6$$

$$a_{12} = 31250 .$$

Second and Higher order Linear Homogeneous Recurrence Relations :—

Consider a method of solving recurrence relations of the form.

$$C_n a_n + C_{n-1} a_{n-1} + C_{n-2} a_{n-2} = 0 \quad \text{for } n \geq 2 \text{ —— (1)}.$$

Where C_n C_{n-1} and C_{n-2} are real constants with $C_n \neq 0$.

A relation of this types is called a second order linear homogeneous recurrence relation with constant co efficients.

Suppose a solution of relation (1) in the form $a_n = ck^n$ where $c \neq 0$ and $k \neq 0$.

Putting $a_n = ck^n$ in (1), we get

$$C_n ck^n + C_{n-1} ck^{n-1} + C_{n-2} ck^{n-2} = 0.$$

$$ck^n \left(C_n + C_{n-1} k^{-1} + C_{n-2} k^{-2}\right) = 0.$$

$$C_n + C_{n-1} k^{-1} + C_{n-2} k^{-2} = 0.$$

$$C_n k^2 + C_{n-1} k + C_{n-2} = 0 \text{ —— (2)}.$$

Thus $a_n = ck^n$ is a solution of (1) if k satisfies the quadratic equation (2).

This quadratic equation is called the auxiliary equation or the characte-
-ristic equation for the relation (1).

Case(i) :- The two roots k_1 and k_2 of equation (2) are real and distinct

Then we take $a_n = Ak_1^n + Bk_2^n$ where A and B are arbitrary real constants, as the general solution of the relation (1).

Case (ii) :- The two roots k_1 and k_2 of equation (2) are real and equal with k as the common value Then we take $a_n = (A + Bn)k^n$ where A and B are arbitrary real constants, as the general solution of the relation (1).

Case (iii) :- The two roots k_1 and k_2 of equation (2) are complex Then k_1 and k_2 are complex conjugate of each other so that if $k_1 = p + iq$ then $k_2 = p - iq$ and we take $a_n = r^n(A \cos n\theta + B \sin n\theta)$ where A and B are

arbitrary complex constants, $r = |k_1| = |k_2| = \sqrt{p^2+q^2}$ and $\theta = \tan^{-1}(q/p)$.
as the general solution of the relation ①.

→ consider a method of solving recurrence relations of the form.

$$c_n a_n + c_{n-1} a_{n-1} + c_{n-2} a_{n-2} + \dots + c_{n-k} a_{n-k} = 0 \quad \text{for } n \geq k \geq 3 \text{ ——— } ③$$

where $c_n, c_{n-1}, \dots c_{n-k}$ are real constants with $c_n \neq 0$.

Which is a linear homogeneous recurrence relation of order "k"

suppose a solution of relation ③ in form $a_n = ck^n$ where $c \neq 0$, $k \neq 0$.

Put $a_n = ck^n$ in ③, we get.

$$c_n ck^n + c_{n-1} ck^{n-1} + c_{n-2} ck^{n-2} + \dots + c_{n-k} ck^{n-k} = 0.$$

$$ck^n \left[c_n + c_{n-1} k^{-1} + c_{n-2} k^{-2} + \dots + c_{n-k} k^{-k} \right] = 0.$$

$$c_n k^k + c_{n-1} k^{k-1} + c_{n-2} k^{k-2} + \dots + c_{n-k} = 0 \text{ ——— } ④$$

Thus, $a_n = ck^n$ is a solution of ① if k satisfies the equation ④.
This equation is called the auxiliary equation or the characteristic equation for the relation ③.

Case (i):- If $k_1, k_2, k_3 \dots k_k$ are roots of the characteristic equation such that $k_i \neq k_j$, $i \neq j$.
Then we take $u_n = A_1 k_1^n + A_2 k_2^n + A_3 k_3^n + \dots + A_k k_k^n$. where $A_1, A_2, \dots A_n$ are arbitrary real constants as the general solution of the relation ③.

case (ii):- If $k_1 = k_2 = k_3$, and the roots $k_4, k_5 \dots k_k$ are distinct.
Then we take $u_n = (A_1 + A_2 n + A_3 n^2) k_1^n + A_4 k_4^n + \dots + A_k k_k^n$.
where $A_1\ A_2\ A_3 \dots A_k$ are arbitrary real constants. as the general solution of the real (1).

Case (iii):- If two complex roots are repeated. and remaining roots are distinct
$k_1 = k_2 = \alpha + i\beta$, $k_3 = k_4 = \alpha - i\beta$

We take $u_n = r^n \left((A_1 + A_2 n)\cos n\theta + (A_3 + A_4 n)\sin n\theta \right) + A_5 k_5^n + \dots + A_k k_k^n$.
$r = |k_1| = \sqrt{\alpha^2+\beta^2}$. where $A_1, A_2 \dots A_k$ are arbitrary complex constants.

Find the general solution of the recurrence relation $a_n + a_{n-3} = 0$, $n \geq 3$.

Sol:- Given that the recurrence relation $a_n + a_{n-3} = 0$, $n \geq 3$ —— ①.

Let the solution of the relation in form $a_n = ck^n$ —— ② where $k \neq 0$, $c \neq 0$

sub. ② in ①, we get

$$ck^n + ck^{n-3} = 0$$

$$ck^n(1 + k^{-3}) = 0$$

$$ck^n \neq 0. \quad k^3 + 1 = 0.$$

∴ The characteristic equation ① is $k^3 + 1 = 0$

$$\Rightarrow (k+1)(k^2 - k + 1) = 0$$

The roots of characteristic eqn are $k = -1, \frac{1}{2}(1 + \sqrt{3}i), \frac{1}{2}(1 - \sqrt{3}i)$.

The roots are real and complex.

The general solution for a_n is $a_n = A(-1)^n + r^n[c_1 \cos n\theta + c_2 \sin n\theta]$.

Where A, c_1, c_2 are arbitrary constants.

$$r = |k_2| = |k_3| = \frac{1}{2}\sqrt{1^2 + (\sqrt{3})^2} = 1.$$

$$\tan\theta = \frac{\sqrt{3}/2}{1/2} = \sqrt{3} \Rightarrow \theta = \frac{\pi}{3}$$

$$\therefore a_n = A(-1)^n + \left[c_1 \cos\frac{n\pi}{3} + c_2 \sin\frac{n\pi}{3}\right].$$

This is the general solution of the given recurrence relation

Find the general solution of the recurrence relation $a_n - 7a_{n-2} + 10a_{n-4} = 0$, $n \geq 4$.

Sol:- Given that the recurrence relation $a_n - 7a_{n-2} + 10a_{n-4} = 0$ $n \geq 4$ —— ①

Let the solution of the relation in the form $a_n = ck^n$ —— ② where $c \neq 0$, $k \neq 0$.

Sub. ② in ①, we get.

$$ck^n - 7ck^{n-2} + 10ck^{n-4} = 0.$$

$$ck^n[1 - 7k^{-2} + 10k^{-4}] = 0.$$

$$ck^n \neq 0. \quad k^4 - 7k^2 + 10 = 0.$$

The characteristic equation of relation ① is $K^4 - 7K^2 + 10 = 0$.

i.e $(K^2)^2 - 7(K^2) + 10 = 0$.

$$\therefore K^2 = \frac{7 \pm \sqrt{49-40}}{2} = \frac{1}{2}(7 \pm 3)$$

i.e $K_1^2 = \frac{7+3}{2} = 5$ and $K_2^2 = \frac{7-3}{2} = 2$

$K_1^2 = 5 \Rightarrow K_1 = \pm\sqrt{5}$

$K_2^2 = 2 \Rightarrow K_2 = \pm\sqrt{2}$

$\therefore$ The roots are real and distinct.

$\therefore$ The general solution for a_n is $a_n = A(\sqrt{5})^n + B(-\sqrt{5})^n + C(\sqrt{2})^n + D(-\sqrt{2})^n$

Where A, B, C, D are arbitrary constants.

Solve the recurrence relation $a_n = a_{n-1} + 2a_{n-2}$ with $a_0 = 2$ and $a_1 = 7$.

Sol: Given that $a_n - a_{n-1} - 2a_{n-2} = 0$ — ①

Which is 2nd order linear homo. recurrence relation.

Let the solution of the relation ① is in form $a_n = cK^n$ — ②

Sub. ② in ①, we get

$cK^n - cK^{n-1} - 2cK^{n-2} = 0$

$cK^n [1 - K^{-1} - 2K^{-2}] = 0$

$cK^n [K^2 - K - 2] = 0$

$cK^n \neq 0$. $K^2 - K - 2 = 0$

Which is characteristic equation of ①.

$$K = \frac{1 \pm \sqrt{1-4(-2)}}{2} = \frac{1 \pm \sqrt{9}}{2} = \frac{1 \pm 3}{2}$$

$K_1 = 2, -1$

The roots are real and distinct

$\therefore$ The general solution of ① is

$a_n = A_1 2^n + A_2 (-1)^n$.

Solve the recurrence relation $a_{n+2}^2 - 5a_{n+1}^2 + 4a_n^2 = 0$ for $n \geq 0$.

given $a_0 = 4$ and $a_1 = 13$.

Sol: Given that $a_{n+2}^2 - 5a_{n+1}^2 + 4a_n^2 = 0$ for $n \geq 0$. —— ①

Which is not linear in a_n.

To make this linear assume $b_n = a_n^2$

Then the original recurrence relation becomes

$$b_{n+2} - 5b_{n+1} + 4b_n = 0 \quad \text{— ②}.$$

Which is 2^{nd} order homogeneous recurrence relation.

Let the solution of the relation in the form $b_n = ck^n$ —— ③ where $k \neq 0$, $c \neq 0$.

Sub. ③ in ②, we get

$$ck^{n+2} - 5ck^{n+1} + 4ck^n = 0.$$

$$ck^n(k^2 - 5k + 4) = 0.$$

$$k^2 - 5k + 4 = 0.$$

$\therefore$ The characteristic equation of relation ② is $k^2 - 5k + 4 = 0$.

$$(k-1)(k-4) = 0$$

$$k = 1, 4.$$

The roots are real and distinct.

$\therefore$ The general solution of the given relation is

$$b_n = A1^n + B4^n$$

$$b_n = A + B4^n \quad \text{—— ④}.$$

Where A and B are arbitrary constants.

Given that $a_0 = 4$ and $a_1 = 13$

$4 = A + B$.

From ④, we get $A + B = 16$.

$A + 4B = 169$.

Solve above eqn's, we get $A = -35$ $B = 51$

$b_n = (-35) + (51)4^n$.

$a_n^2 = (51)4^n - 35 \Rightarrow a_n = \pm\sqrt{(51)4^n - 35}$ $(\because b_n = a_n^2)$

Which is the required solution of given relation

Solve the recurrence relation. $a_{n+2} = a_{n+1} + a_n$ for $n \geq 0$. given $a_0 = 0, a_1 = 1$.

Sol: Given that $a_{n+2} - a_{n+1} - a_n = 0$. for $n \geq 0$. —①

Let the solution of the relation in the form $a_n = ck^n$ —② where $c \neq 0, k \neq 0$.

Sub. ② in ①, we get.

$$ck^{n+2} - ck^{n+1} - ck^n = 0.$$

$$ck^n (k^2 - k - 1) = 0.$$

$$k^2 - k - 1 = 0$$

∴ The characteristic equation of given relation is $k^2 - k - 1 = 0$

$$k = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1}{2}(1 \pm \sqrt{5}).$$

The roots are real and distinct.

The general solution of given relation is.

$$a_n = A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n \quad \text{—③}.$$

Where A and B are arbitrary constants.

We have $a_0 = 0, a_1 = 1$.

From ③. we get $\quad 0 = A\left(\frac{1+\sqrt{5}}{2}\right)^0 + B\left(\frac{1-\sqrt{5}}{2}\right)^0 = A + B$

$$A + B = 0$$

$$1 = A\left(\frac{1+\sqrt{5}}{2}\right) + B\left(\frac{1-\sqrt{5}}{2}\right)$$

Solving above eqn's, we get.

$$A = -B = \frac{1}{\sqrt{5}}.$$

$$\therefore \quad a_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

(1) Solve the recurrence relation $a_n + a_{n-1} - 6a_{n-2} = 0$ for $n \geq 2$. given that $a_0 = -1$ and $a_1 = 8$.

Sol:- Given that the recurrence relation $a_n + a_{n-1} - 6a_{n-2} = 0$ for $n \geq 2$. —— (1)

Let the solution of the relation in the form $a_n = ck^n$ where $c \neq 0, k \neq 0$. —— (2)

Sub. (2) in (1), we get.

$$ck^n + ck^{n-1} - 6ck^{n-2} = 0.$$

$$ck^n[1 + k^{-1} - 6k^{-2}] = 0.$$

$$ck^n[k^2 + k - 6] = 0.$$

$$k^2 + k - 6 = 0.$$

∴ The characteristic equation is $k^2 + k - 6 = 0$

$$(k+3)(k-2) = 0$$

$$k = 2, -3.$$

The roots of characteristic equation are $k_1 = 2$, $k_2 = -3$.

Which are real and distinct.

∴ The general solution of the given relation is.

$$a_n = A2^n + B(-3)^n. \quad \text{—— (3)}$$

Where A and B are arbitrary constants.

Given that $a_0 = -1$, $a_1 = 8$.

From (3), we get $-1 = A + B$.

$$8 = -3A + 2B.$$

Solving these, we get $A = -2$ and $B = 1$.

$$\therefore a_n = (-2)2^n + (-3)^n.$$

$$a_n = -2^{n+1} + (-3)^n.$$

This is the solution of the given relation, under the given initial. conditions $a_0 = -1$ and $a_1 = 8$.

Solve the recurrence relation $a_n = 2(a_{n-1} - a_{n-2})$ for $n \geq 2$, given that $a_0 = 1$ and $a_1 = 2$.

Sol:- Given that the recurrence relation $a_n = 2(a_{n-1} - a_{n-2})$ for $n \geq 2$. —— (1)

Let the solution of the relation in the form $a_n = ck^n$ —— (2) where $c \neq 0$, $k \neq 0$

Sub. (2) in (1), we get

$$ck^n - 2ck^{n-1} + 2ck^{n-2} = 0.$$

$$ck^n[1 - 2k^{-1} + 2k^{-2}] = 0$$

$$k^2 - 2k + 2 = 0.$$

The characteristic equation of relation (1) is $k^2 - 2k + 2 = 0$

$$k = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$$

$\therefore$ The roots $k_1 = 1+i$, $k_2 = 1-i$ are complex.

$\therefore$ The general solution of the given relation is.

$$a_n = r^n[A\cos n\theta + B\sin n\theta]$$

Where A and B are arbitrary constants $r = |(1 \pm i)| = \sqrt{2}$ and $\theta = \tan^{-1}\left(\frac{1}{1}\right) \Rightarrow$

$\Rightarrow \theta = \frac{\pi}{4}$.

$$a_n = (\sqrt{2})^n\left[A\cos\frac{n\pi}{4} + B\sin\frac{n\pi}{4}\right] \quad \text{—— (3)}$$

Given that $a_0 = 1$ $a_1 = 2$

From (3) we get, $1 = A$

$$2 = \sqrt{2}\left[A\cos\frac{\pi}{4} + B\sin\frac{\pi}{4}\right] = A + B$$

$$2 = A + B.$$

$$\therefore B = 1.$$

$$\therefore a_n = (\sqrt{2})^n\left[\cos\frac{n\pi}{4} + \sin\frac{n\pi}{4}\right].$$

This is the solution of the given relation under the initial conditions $a_0 = 1$, $a_1 = 2$.

Non homogeneous Recurrence Relations of second and higher orders

The general form of the higher order linear non homogeneous recurrence relations with constant coefficients is.

$$c_n a_n + c_{n-1} a_{n-1} + c_{n-2} a_{n-2} + \dots + c_{n-k} a_{n-k} = f(n) \quad \text{for } n \geq k \geq 2 \quad \text{—— (1)}$$

Where $c_n, c_{n-1}, c_{n-2}, \dots c_{n-k}$ are real constants with $c_n \neq 0$ and $f(n)$ is a given real valued function of n.

A general solution of the recurrence relation (1) is given by.

$$a_n = a_n^{(h)} + a_n^{(p)} \quad \text{—— (2)}$$

Where $a_n^{(h)}$ is the general solution of the homogeneous part of the relation (1), namely the relation (1) with $f(n)=0$. and $a_n^{(p)}$ is any particular solution of the relation (1).

Case (i):- Suppose $f(n)$ is a polynomial of degree q and 1 is not a root of the characteristic equation of the homogeneous part of the relation (1). In this case $a_n^{(p)}$ is taken in the form.

$$a_n^{(p)} = A_0 + A_1 n + A_2 n^2 + \dots + A_q n^q \quad \text{—— (3)}$$

Where $A_0, A_1, A_2, \dots A_q$ are constants to be evaluated by using the fact that $a_n = a_n^{(p)}$ satisfies the relation (1).

Case (ii):- Suppose $f(n)$ is a polynomial of degree q and 1 is a root of multiplicity m of the characteristic equation of the homogeneous part of the relation (1). In this case $a_n^{(p)}$ is taken in the form.

$$a_n^{(p)} = n^m \left[A_0 + A_1 n + A_2 n^2 + \dots A_q n^q \right].$$

Where $A_0, A_1, A_2, \dots A_q$ are constants to be evaluated by using the fact that $a_n = a_n^{(p)}$ satisfies the relation (1).

Case (iii) :- Suppose $f(n) = ab^n$, where a is a constant and b is not a root of the characteristic equation of the homogeneous part of the relation ①.

Then $a_n^{(P)}$ is taken in the form $a_n^{(P)} = A_0 b^n$ —— ⑤.

where A_0 is a constant to be evaluated by using the fact that $a_n = a_n^{(P)}$ satisfies the relation ①.

Case (iv) :— Suppose $f(n) = ab^n$ where a is a constant and b is a root of multiplicity m of the characteristic equation of the homogeneous part of the relation ①. Then $a_n^{(P)}$ is taken in the form.

$$a_n^{(P)} = A_0 n^m b^n \quad —— ⑥.$$

Where A_0 is a constant to be evaluated by using the fact that $a_n = a_n^{(P)}$ satisfies the relation ①.

Case (v) :- Suppose $f(n) = b^n \phi(n)$ and if b is a root of characteristic equation of multiplicity m. Then $a_n^{(P)}$ is taken in the form.

$$a_n^{(P)} = n^m (A_0 + A_1 n + A_2 n^2 + \cdots + A_q n^q) . b^n.$$

Where $A_0, A_1, A_2 \ldots A_q$ are constants to be evaluated by using the fact that $a_n = a_n^{(P)}$ satisfies the relation ①.

Case (vi) :- Suppose $f(n) = b^n \phi(n)$ and if b is not root of characteristic equation. Then $a_n^{(P)}$ is taken in the form.

$$a_n^{(P)} = b^n (A_0 + A_1 n + A_2 n^2 + \cdots + A_q n^q)$$

Where $A_0, A_1, A_2 \ldots A_q$ are constants to be evaluated by using the fact that $a_n = a_n^{(P)}$ satisfies the relation ①.

Case (vii) :- Suppose $f(n) = c \sin n\theta$ where c is a constant. Then $a_n^{(P)}$ is taken in the form. $a_n^{(P)} = A_1 \cos n\theta + A_2 \sin n\theta$.

Where A_1, A_2 are constants to be evaluated by using the fact that $a_n = a_n^{(P)}$ satisfies the relation ①.

Case (viii) :- suppose $f(n) = c \cos n\theta$ where c is a constant. Then $a_n^{(p)}$ is taken in the form $a_n^{(p)} = A_1 \cos n\theta + A_2 \sin n\theta$. Where A_1 A_2 are constants to be evaluated by using the fact that $a_n = a_n^{(p)}$ satisfies the relation ①.

Case (ix) :- suppose $f(n) = c\gamma^n \sin n\theta$ (or) $c\gamma^n \cos n\theta$ where c is a constant. Then $a_n^{(p)}$ is taken in the form $a_n^{(p)} = A_1 \gamma^n \cos n\theta + A_2 \gamma^n \sin n\theta$. Where A_1 A_2 are constants to be evaluated by using the fact that $a_n = a_n^{(p)}$ satisfies the relation ①.

Case (x) :- suppose $f(n) = c\gamma^n \cos n\theta$ (or) $c\gamma^n \sin n\theta$ where c is a constant and $c\gamma^n \cos n\theta$ (or) $c\gamma^n \sin n\theta$ is root of the characteristic equation of multiplicity m then $a_n^{(p)}$ is taken in the form.

$$a_n^{(p)} = n^m (A_1 \gamma^n \cos n\theta + A_2 \gamma^n \sin n\theta)$$

where A_1 A_2 are constants to be evaluated by using the fact that $a_n = a_n^{(p)}$ satisfies the relation ①.

(19)

Solve the recurrence relation $a_{n+2} - 2a_{n+1} + a_n = 3n+5$; given $a_0 = 1, a_1 = 1$

Sol:- Given that the relation $a_{n+2} - 2a_{n+1} + a_n = 3n+5$ — ①.

The homogeneous part of the relation ① is $a_{n+2} - 2a_{n+1} + a_n = 0$ — ②

Let the solution of the relation ② is in the form $a_n = ck^n$ — ③.

where $c \neq 0$ $k \neq 0$

Sub. ③ in ②, we get

$$ck^{n+2} - 2ck^{n+1} + ck^n = 0.$$

$$ck^n(k^2 - 2k + 1) = 0.$$

$$ck^n \neq 0. \quad k^2 - 2k + 1 = 0$$

Which is the characteristic equation of relation ①

$$(k-1)^2 = 0$$

$$k = 1, 1$$

The roots are real and repeated.

$\therefore$ The general solution of relation ② is $a_n^{(h)} = A + Bn$.

We observe that R.H.S of relation ① is a polynomial of degree 1

and 1 is the root of char. equation of multiplicity 2.

Let us take $a_n^{(P)} = n^m(A_1 + A_2 n)$

$$a_n^{(P)} = n^2(A_1 + A_2 n) \quad — ④.$$

Sub. ④ in ①, we get.

$$(n+2)^2(A_1 + A_2(n+2)) - 2(n+1)^2(A_1 + A_2(n+1)) + n^2(A_1 + A_2 n) = 3n+5$$

$$A_1(n+2)^2 + A_2(n+2)^3 - 2A_1(n+1)^2 - 2A_2(n+1)^3 + A_1 n^2 + A_2 n^3 = 3n+5$$

Compare the Co efficients of powers of n both sides, we get.

$$A_2 - 2A_2 + A_2 = 0.$$

$$A_1 + 6A_2 - 2A_1 - 6A_2 + A_1 = 0.$$

$$4A_1 + 12A_2 - 4A_1 - 6A_2 = 3$$

$$A_2 = \frac{1}{2}.$$

$$4A_1 + 8A_2 - 2A_1 - 2A_2 = 5$$

$$2A_1 + 6A_2 = 5$$

$$A_1 = 1.$$

$\therefore \; a_n^{(p)} = n^2\left(1+\frac{1}{2}n\right).$

$\therefore$ The general solution of ① is $a_n = a_n^{(h)} + a_n^{(p)}$

$$a_n = (A+Bn) + n^2\left(1+\frac{n}{2}\right) \quad \text{—⑤}$$

We have $a_0 = 1 \;\; a_1 = 1$ —— 8

From ⑤, $\quad a_0 = A \Rightarrow A = 1$

$$a_1 = (A+B) + \left(1+\frac{1}{2}\right) \Rightarrow a_1 = A+B+\frac{3}{2}$$

$$A+B = 1-\frac{3}{2} = -\frac{1}{2}$$

$$A+B = -\frac{1}{2}$$

$$B = -\frac{3}{2}$$

$$\therefore \; a_n = \left(1-\frac{3}{2}n\right) + n^2\left(1+\frac{n}{2}\right).$$

Find a general expression for a solution to the recurrence relation.

$a_n - 5a_{n-1} + 6a_{n-2} = n(n-1)$ for $n \geq 2$.

Sol: Given that the relation $a_n - 5a_{n-1} + 6a_{n-2} = n^2 - n$ —①.

The homogeneous part of the relation ① is $a_n - 5a_{n-1} + 6a_{n-2} = 0$ —②.

Let the solution of the relation ② is in the form $a_n = c\,k^n$ —③

sub. ③ in ②, we get

$$ck^n - 5ck^{n-1} + 6ck^{n-2} = 0$$

$$ck^n\left(1 - 5k^{-1} + 6k^{-2}\right) = 0$$

$$ck^n\left(k^2 - 5k + 6\right) = 0.$$

$$ck^n \neq 0 \quad k^2 - 5k + 6 = 0.$$

Which is the characteristic equation of the relation ②.

$$(k-2)(k-3) = 0$$

$$k = 2, 3.$$

The roots are real and distinct.

$\therefore$ The general solution of relation ② is $a_n^{(h)} = Ak_1^n + Bk_2^n$

i.e $a_n^{(h)} = A\,2^n + B\,3^n$.

(20)

We observe that the R.H.S of eqn ① is a polynomial of degree 2.

Let us take $a_n^{(p)} = A_1 + A_2 n + A_3 n^2$.

$a_{n-1}^{(p)} = A_1 + A_2(n-1) + A_3(n-1)^2$

$a_{n-2}^{(p)} = A_1 + A_2(n-2) + A_3(n-2)^2$.

Sub. these values in eqn ①, we get.

$(A_1 + A_2 n + A_3 n^2) - 5(A_1 + A_2(n-1) + A_3(n-1)^2) + 6(A_1 + A_2(n-2) + A_3(n-2)^2)$
$= n^2 - n$

Comparing the coefficients of powers of n bothsides.

$A_3 - 5A_3 + 6A_3 = 1 \Rightarrow A_3 = \frac{1}{2}$.

$A_2 - 5A_2 + 10A_3 + 6A_2 - 24A_3 = -1$.

$2A_2 - 14A_3 = -1$.

$2A_2 = -1 + 14A_3 = 6$.

$A_2 = 3$.

$A_1 - 5A_1 + 5A_2 + 10A_3 + 6A_1 - 12A_2 + 24A_3 = 0$.

$2A_1 - 7A_2 + 34A_3 = 0$.

$2A_1 - 21 + 17 = 0$.

$A_1 = 2$.

$\therefore a_n^{(p)} = 2 + 3n + \frac{1}{2}n^2$.

$\therefore$ The general solution of given recurrence relation is

$a_n = a_n^{(h)} + a_n^{(p)}$

$a_n = A2^n + B3^n + (2 + 3n + \frac{1}{2}n^2)$

Solve the recurrence relation. $a_n - 5a_{n-1} + 6a_{n-2} = 1$.

Sol:- Given that the relation $a_n - 5a_{n-1} + 6a_{n-2} = 1$ —— ①.

which is 2^{nd} order linear non homo. recurrence relation.

The homogeneous part of the relation ① is $a_n - 5a_{n-1} + 6a_{n-2} = 0$ —— ②.

Let the solution of the relation ② is in the form $a_n = CK^n$ — ③

where $C \neq 0$ $K \neq 0$.

Sub. ③ in ②, we get

$$CK^n - 5CK^{n-1} + 6CK^{n-2} = 0.$$

$$CK^n[1 - 5K^{-1} + 6K^{-2}] = 0$$

$$CK^n[K^2 - 5K + 6] = 0$$

$$CK^n \neq 0 \quad K^2 - 5K + 6 = 0.$$

Which is the characteristic equation of relation ②.

$(K-3)(K-2) = 0$. i.e $K = 2, 3$.

The roots of characteristic equation are 2, 3.

The roots are real and distinct.

∴ The general solution of relation ② is $a_n^{(h)} = AK_1^n + BK_2^n$

i.e $a_n^{(h)} = A2^n + B3^n$.

We observe that the R.H.S of eqn relation ① is a polynomial of degree 0.

Let us take $a_n^{(p)} = A_0$.

$$a_{n-1}^{(p)} = A_0 \quad a_{n-2}^{(p)} = A_0.$$

Sub. these values in eqn ①, we get.

$$A_0 - 5A_0 + 6A_0 = 1.$$

$$2A_0 = 1$$

$$A_0 = \frac{1}{2}.$$

$$\therefore a_n^{(p)} = \frac{1}{2}.$$

∴ The general solution of ① is $a_n = a_n^{(h)} + a_n^{(p)}$.

$$\therefore a_n = A2^n + B3^n + \frac{1}{2}.$$

(21)

Solve the recurrence relation. $a_n + 2a_{n-1} - 3a_{n-2} = 4n^2 - 5$, for $n \geq 2$.

Sol:- Given that the recurrence relation $a_n + 2a_{n-1} - 3a_{n-2} = 4n^2 - 5$, for $n \geq 2$. ——(1).

The homogeneous form of given relation is $a_n + 2a_{n-1} - 3a_{n-2} = 0$ ——(2).

Let the solution of the relation (2) in the form $a_n = ck^n$ ——(3)

Sub. (3) in (2), we get $ck^n + 2ck^{n-1} - 3ck^{n-2} = 0$

$$ck^n [1 + 2k^{-1} - 3k^{-2}] = 0$$

$$ck^n [k^2 + 2k - 3] = 0.$$

$$ck^n \neq 0, \quad k^2 + 2k - 3 = 0.$$

The characteristic equation of relation (2) is $k^2 + 2k - 3 = 0$

$$(k-1)(k+3) = 0$$

The roots are real & distinct. $k = 1, -3$.

$\therefore$ The general solution of (2) is $a_n^{(h)} = A.1^n + B(-3)^n$. ——(4)

Where A, B are arbitrary constants.

Since 1 is a simple root of the characteristic equation. and the R.H.S of the given relation is a polynomial of degree 2.

$$a_n^{(P)} = n^1(A_0 + A_1 n + A_2 n^2)$$

$$a_n^{(P)} = A_0 n + A_1 n^2 + A_2 n^3. \quad \text{—— (5)}$$

Where A_0, A_1, A_2 are constants.

Putting (5) for a_n in the given relation, we get

$$(A_0 n + A_1 n^2 + A_2 n^3) + 2\{A_0 n + A_1 (n-1)^2 + A_2 (n-1)^3\} - 3\{A_0(n-2) + A_1(n-2)^2 + A_2(n-2)^3\} = 4n^2 - 5.$$

Equating the corresponding terms on the two sides, we get.

$$A_2 + 2A_2 - 3A_2 = 0.$$

$$A_1 + 2A_1 - 6A_2 - 3A_1 + 18A_2 = 4.$$

$$A_0 + 2A_0 - 4A_1 + 6A_2 - 3A_0 + 12A_1 - 36A_2 = 0.$$

$$-2A_0 + 2A_1 - 2A_2 + 6A_0 - 12A_1 + 24A_2 = -5$$

These gives $\quad 12A_2 = 4 \quad \Rightarrow A_2 = 1/3.$

$8A_1 - 20A_2 = 0 \quad \Rightarrow A_1 = 5/6.$

$4A_0 - 10A_1 + 22A_2 = 0 \quad \Rightarrow$ So that $A_0 = 1/4.$

Sub. the values of A_0, A_1, A_2 in (5), we get

$$a_n^{(P)} = n\left(\frac{1}{4} + \frac{5n}{6} + \frac{1}{3}n^2\right).$$

$\therefore$ The general solution for a_n is $\quad a_n = a_n^{(h)} + a_n^{(P)}$

$$a_n = A + B(-3)^n + n\left(\frac{1}{4} + \frac{5}{6}n + \frac{1}{3}n^2\right).$$

Solve the recurrence relation. $a_n + 4a_{n-1} + 4a_{n-2} = 5\times(-2)^n, \quad n \geq 2.$

Sol: Given that the recurrence relation $a_n + 4a_{n-1} + 4a_{n-2} = 5\times(-2)^n, n \geq 2.$ —— (1).

The homogeneous form of the given relation is $a_n + 4a_{n-1} + 4a_{n-2} = 0$ —— (2).

Let $a_n = ck^n$ (3) is the solution of the relation (2). Where $c \neq 0, k \neq 0.$

Sub. (3) in (2), we get

$$ck^n + 4ck^{n-1} + 4ck^{n-2} = 0.$$

$$ck^n\left[1 + 4k^{-1} + 4k^{-2}\right] = 0$$

$$ck^n \neq 0 \quad k^2 + 4k + 4 = 0.$$

The characteristic equation of the relation (2) is $k^2 + 4k + 4 = 0$

$(k+2)^2 = 0$ i.e $k = -2, -2$

The roots are real and repeated.

$\therefore \quad a_n^{(h)} = (A + Bn)(-2)^n.$ Where A, B are arbitrary Constants.

We observe that the RHS of the given relation contains $(-2)^n$ as a factor and -2 is a repeated root of the characteristic equation.

$$a_n^{(P)} = A_0 n^2 (-2)^n$$

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Putting this for a_n in the given relation, we get

$$A_0 n^2 (-2)^n + 4A_0 (n-1)^2 (-2)^{n-1} + 4A_0 (n-2)^2 (-2)^{n-2} = 5(-2)^n .$$

Divide above equation by $(-2)^{n-2}$, we get

$$A_0 \left[n^2 (-2)^2 + 4(n-1)^2 (-2) + 4(n-2)^2\right] = 5(-2)^2 .$$

$$A_0 \left\{4n^2 - 8(n^2-2n+1) + 4(n^2-4n+4)\right\} = 20$$

$$A_0 \left[\cancel{4n^2} - \cancel{8n^2} + \cancel{16n} - 8 - \cancel{16n} + 16\right] = 20$$

$$A_0 = \frac{20}{8} = \frac{5}{2} .$$

$$\therefore\ a_n^{(P)} = \frac{5}{2} n^2 (-2)^n .$$

$\therefore$ The general solution of the given relation is $a_n = a_n^{(h)} + a_n^{(P)}$

$$a_n = (A+Bn)(-2)^n + \frac{5}{2} n^2 (-2)^n .$$

$$a_n = \left(A + Bn + \frac{5}{2} n^2\right)(-2)^n .$$

Solve the recurrence relation $a_{n+2} - 6a_{n+1} + 9a_n = 3\times 2^n + 7\times 3^n$ for $n \geq 0$. given $a_0 = 1,\ a_1 = 4$. — ①

Sol:- Given that the recurrence relation $a_{n+2} - 6a_{n+1} + 9a_n = 3\times 2^n + 7\times 3^n$ for $n \geq 0$.

The homogeneous form of the given relation is $a_{n+2} - 6a_{n+1} + 9a_n = 0$ — ②.

Let the solution of the relation ② in the form $a_n = ck^n$ — ③ where $c \neq 0$, $k \neq 0$.

Sub. ③ in ②, we get

$$ck^{n+2} - 6ck^{n+1} + 9ck^n = 0 .$$

$$ck^n (k^2 - 6k + 9) = 0$$

$$ck^n \neq 0 \quad k^2 - 6k + 9 = 0$$

The characteristic equation of relation ② is $k^2 - 6k + 9 = 0$ i.e $(k-3)^2 = 0$

$$k = 3, 3 .$$

The roots are real and repeated.

The general solution of ② is $a_n^{(h)} = (A+Bn)\, 3^n$ — ④

Where A, B are arbitrary constants.

We observe that the R.H.S of the given relation contains 3^n and 3 is a repeated root of the characteristic equation.

$$a_n^{(p)} = C\,2^n + D.n^2 \times 3^n \quad \text{—} (5)$$

[illegible] constants.

Putting this for a_n in the given relation, we [illegible]

$$\{C.2^{n+2} + D(n+2)^2 3^{n+2}\} - 6\{C\,2^{n+1} + D(n+1)^2 . 3^{n+1}\} + 9\{C2^n + Dn^2 \times 3^n\} = 3.2^n + 7.3^n.$$

Equating the corresponding terms on bothsides, we get.

$$C.2^2 - 6C\times 2 + 9C = 3 \quad \text{and} \quad D(n+2)^2 \times 3^2 - 6D(n+1)^2 \times 3 + 9Dn^2 = 7.$$

These give $C = 3$ and $D = \dfrac{7}{9(n+2)^2 - 18(n+1)^2 + 9n^2} = \dfrac{7}{18}$.

Sub. the values of C and D in (5), weget

$$a_n^{(p)} = 3\,2^n + \frac{7n^2}{18} 3^n.$$

$\therefore$ The general solution of given relation is $a_n = a_n^{(h)} + a_n^{(p)}$

$$a_n = \left(A + Bn + \frac{7}{18}n^2\right) \times 3^n + 3\times 2^n \quad \text{—} (6).$$

Given that $a_0 = 1$, $a_1 = 4$

From (6), weget $1 = A + 3 \implies A = -2$.

$$4 = \left(A + B + \frac{7}{18}\right) \times 3 + (3\times 2).$$

$$B = \frac{17}{18}.$$

$$\therefore a_n = \left(\frac{7}{18}n^2 + \frac{17}{18}n - 2\right) \times 3^n + 3\times 2^n.$$

Solve the recurrence relation $a_n + 5a_{n-1} + 6a_{n-2} = 42(4)^n$.

Sol:- Given that the recurrence relation $a_n + 5a_{n-1} + 6a_{n-2} = 42(4)^n$ for $n \geq 2$ — (1)

The homogeneous form of given relation is $a_n + 5a_{n-1} + 6a_{n-2} = 0$ — (2).

Let the solution of the relation (2) in the form $a_n = ck^n$ — (3).

sub. (3) in (2), we get $ck^n + 5ck^{n-1} + 6ck^{n-2} = 0$.

$$ck^n\left[1 + 5k^{-1} + 6k^{-2}\right] = 0.$$

$$ck^n\left[k^2 + 5k + 6\right] = 0.$$

$$ck^n \neq 0 \quad k^2 + 5k + 6 = 0.$$

This is the characteristic equation of relation (2).

$$(k+2)(k+3) = 0$$

$$k = -2, -3.$$

The roots are real and distinct.

$\therefore$ The general solution of (2) is $a_n^{(h)} = A_1(-2)^n + A_2(-3)^n$.

Where A_1, A_2 are arbitrary constants.

Let $a_n^{(p)} = A\,4^n$

$$a_{n-1}^{(p)} = A\,4^{n-1} \quad a_{n-2}^{(p)} = A\,4^{n-2}$$

Sub. these values in the recurrence relation.

$$A4^n + 5A4^{n-1} + 6A4^{n-2} = 42.4^n.$$

$$A4^n\left(1 + 54^{-1} + 64^{-2}\right) = 42\,A^n$$

$$A4^n\left(1 + \frac{5}{4} + \frac{6}{16}\right) = 42.4^n.$$

$$A\left(\frac{16 + 20 + 6}{16}\right) = 42$$

$$A = 16.$$

$$\therefore a_n^{(p)} = 16\,4^n.$$

$\therefore$ The general solution of (1) is $a_n = a_n^{(h)} + a_n^{(p)}$

i.e $a_n = A_1(-2)^n + A_2(-3)^n + 16(4)^n$

Solve the recurrence relation $a_{n+2} + 4a_n = 6\cos(\frac{n\pi}{2}) + 3\sin(\frac{n\pi}{2})$.

Sol:- The given recurrence relation is

$$a_{n+2} + 4a_n = 6\cos(\tfrac{n\pi}{2}) + 3\sin(\tfrac{n\pi}{2}) \quad \text{--- (1)}$$

The homogeneous part of the relation (1) is $a_{n+2} + 4a_n = 0$ --- (2)

Let the solution of the relation (2) is in the form $a_n = ck^n$ --- (3)

where $c \neq 0$, $k \neq 0$.

Sub. (3) in (2), we get.

$$ck^{n+2} + 4ck^n = 0$$

$$ck^n (k^2 + 4) = 0$$

$$ck^n \neq 0 \quad k^2 + 4 = 0$$

Which characteristic eqn of (2)

$$(k+2i)(k-2i)$$

$$k = 2i, -2i$$

The roots are imaginary.

The general solution of the relation (2) is $a_n^{(h)} = r^n [A\cos n\theta + B\sin n\theta]$

where A, B are constants $r = |2i| = 2$ $\theta = \tan^{-1}(\frac{2}{0}) = \tan^{-1}(\tan\frac{\pi}{2}) = \frac{\pi}{2}$

$$\theta = \frac{\pi}{2}$$

$$\therefore a_n^{(h)} = 2^n \left[A\cos\frac{n\pi}{2} + B\sin\frac{n\pi}{2}\right].$$

~~Suppose~~ $a_n^{(p)}$ = We observe that R.H.S of eqn (1) is of the form $c\sin n\theta$ (or) $c\cos n\theta$.

We take $a_n^{(p)}$ is in the form $a_n^{(p)} = A_1\cos\frac{n\pi}{2} + A_2\sin\frac{n\pi}{2}$

$$a_{n+1}^{(p)} = A_1\cos(n+1)\frac{\pi}{2} + A_2\sin(n+1)\frac{\pi}{2}$$

$$a_{n+1}^{(p)} = -A_1\sin\frac{n\pi}{2} + A_2\cos\frac{n\pi}{2}.$$

$$a_{n+2}^{(p)} = A_1\cos(n+2)\frac{\pi}{2} + A_2\sin(n+2)\frac{\pi}{2}$$

$$= -A_1\cos\frac{n\pi}{2} + A_2\sin\frac{n\pi}{2}$$

We observe that R.H.S of equation (1) is of the form $c \cos n\theta$.

We take $a_n^{(p)}$ is in the form

$$a_n^{(p)} = A \cos\left(\frac{3n\pi}{4}\right) + B \sin\left(\frac{3n\pi}{4}\right).$$

$$a_{n+1}^{(p)} = A \cos\left(\frac{3\pi(n+1)}{4}\right) + B \sin\left(\frac{3\pi(n+1)}{4}\right)$$

$$= \frac{A}{\sqrt{2}} \cos\left(\frac{3\pi n}{4}\right) - \frac{A}{\sqrt{2}} \sin\left(\frac{3\pi n}{4}\right) - \frac{B}{\sqrt{2}} \sin\frac{3\pi n}{4} + \frac{B}{\sqrt{2}} \cos\frac{3\pi n}{4}.$$

$$a_{n+2}^{(p)} = A_1 \cos\frac{3\pi(n+2)}{4} + A_2 \sin\frac{3\pi(n+2)}{4}.$$

$$a_{n+2}^{(p)} = A_1 \sin\left(\frac{3\pi n}{4}\right) - A_2 \sin\left(\frac{3\pi n}{4}\right)$$

Sub. these values in (1), we get

$$\left[A_1 \sin\left(\frac{3n\pi}{4}\right) - A_2 \sin\left(\frac{3n\pi}{4}\right)\right] + 2\left[\frac{A_1}{\sqrt{2}} \cos\left(\frac{3n\pi}{4}\right) - \frac{A_1}{\sqrt{2}} \sin\left(\frac{3n\pi}{4}\right) - \frac{A_2}{\sqrt{2}} \sin\left(\frac{3n\pi}{4}\right) + \frac{A_2}{\sqrt{2}} \cos\left(\frac{3n\pi}{4}\right)\right] + 2\left[A_1 \cos\left(\frac{3n\pi}{4}\right) + A_2 \sin\left(\frac{3n\pi}{4}\right)\right] = \cos\left(\frac{3n\pi}{4}\right)$$

Comparing the corresponding terms both sides, we get

$$A_1 - A_2 - \sqrt{2}A_1 - \sqrt{2}A_2 + 2A_2 = 0.$$

$$(1-\sqrt{2})A_1 + (1-\sqrt{2})A_2 = 0. \implies A_1 + A_2 = 0 \quad \text{—(a)}$$

$$\sqrt{2}A_1 + \sqrt{2}A_2 + 2A_1 = 1.$$

$$2A_1 + 2\sqrt{2}A_2 = 1 \implies 2(A_1 + \sqrt{2}A_2) = 1$$

$$A_1 + \sqrt{2}A_2 = \frac{1}{2} \quad \text{—(b)}$$

Solving (a) and (b), we get

$$A_2(1-\sqrt{2}) = -\frac{1}{2}$$

$$A_2 = \frac{1}{2(\sqrt{2}-1)}$$

$$A_1 = \frac{1}{2(1-\sqrt{2})}$$

$$\therefore a_n^{(p)} = \frac{1}{2(1-\sqrt{2})}\left[\cos\left(\frac{3n\pi}{4}\right) - \sin\left(\frac{3n\pi}{4}\right)\right].$$

$\therefore$ The general solution of (1) is $a_n = a_n^{(h)} + a_n^{(p)}$

$$\therefore a_n = 2^{\frac{n}{2}}\left[A_1 \cos\left(\frac{3n\pi}{4}\right) + A_2 \sin\left(\frac{3n\pi}{4}\right)\right] + \frac{1}{2(1-\sqrt{2})}\left[\cos\left(\frac{3n\pi}{4}\right) - \sin\left(\frac{3n\pi}{4}\right)\right]$$

sub. these values in (1), we get

$$-A_1\cos\left(\frac{n\pi}{2}\right) - A_2\sin\left(\frac{n\pi}{2}\right) + 4A_1\cos\left(\frac{n\pi}{2}\right) + 4A_2\sin\left(\frac{n\pi}{2}\right) = 6\cos\left(\frac{n\pi}{2}\right) + 3\sin\left(\frac{n\pi}{2}\right)$$

Comparing the coefficients of $\cos\left(\frac{n\pi}{2}\right)$ and $\sin\left(\frac{n\pi}{2}\right)$ bothsides.

$$3A_1 = 6 \Rightarrow A_1 = 2 \qquad 3A_2 = 3 \Rightarrow A_2 = 1.$$

$$\therefore a_n^{(p)} = 2\cos\left(\frac{n\pi}{2}\right) + \sin\left(\frac{n\pi}{2}\right).$$

$\therefore$ The general solution of (1) is $a_n = a_n^{(h)} + a_n^{(p)}$

$$a_n = 2^n\left[A_1\cos\left(\frac{n\pi}{2}\right) + A_2\sin\left(\frac{n\pi}{2}\right)\right] + 2\cos\left(\frac{n\pi}{2}\right) + \sin\left(\frac{n\pi}{2}\right)$$

Solve the recurrence relation $a_{n+2} + 2a_{n+1} + 2a_n = \cos\left(\frac{3n\pi}{4}\right)$.

Sol:- Given that $a_{n+2} + 2a_{n+1} + 2a_n = \cos\left(\frac{3n\pi}{4}\right)$. — (1)

The homogeneous part of the relation (1) is $a_{n+2} + 2a_{n+1} + 2a_n = 0$. (2)

Let the solution of the relation (1) is in the form $a_n = ck^n$ — (3) where $c \neq 0$, $k \neq 0$.

Sub. (3) in (2), we get

$$ck^{n+2} + 2ck^{n+1} + 2ck^n = 0.$$

$$ck^n(k^2 + 2k + 2) = 0$$

$$ck^n \neq 0 \qquad k^2 + 2k + 2 = 0$$

which is the characteristic equation of relation (1).

$$k = \frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i$$

$$|r| = |-1+i| = \sqrt{2} = 2^{1/2}.$$

$$\theta = \tan^{-1}\left(\frac{q}{p}\right) = \tan^{-1}(-1) = \tan^{-1}\left(\tan\frac{3\pi}{4}\right)$$

$$\theta = \frac{3\pi}{4},$$

$\therefore$ The general solution of (2) is $a_n^{(h)} = 2^{\frac{n}{2}}\left(A_1\cos\left(\frac{3\pi}{4}\right) + A_2\sin\left(\frac{3\pi}{4}\right)\right)$

Solve the recurrence relation $a_{n+2}+4a_n = 2^n \cos\left(\frac{n\pi}{2}\right)$.

Sol: Given that $a_{n+2}+4a_n = 2^n \cos\left(\frac{n\pi}{2}\right)$ ——(1)

The homogeneous part of the relation (1) is $a_{n+2}+4a_n=0$ ——(2)

Let the solution of relation (2) is in the form $a_n = ck^n$ ——(3)

where $c\neq 0$ $k\neq 0$.

Sub. (3) in (2), we get

$$ck^{n+2}+4ck^n=0$$

$$ck^n(k^2+4)=0$$

$$ck^n\neq 0 \quad k^2+4=0.$$

Which is the characteristic equation of (1).

$$k=\pm 2i$$

The roots are imaginary.

The general solution (1) is $a_n^{(h)} = 2^n\left[A_1\cos\left(\frac{n\pi}{2}\right)+A_2\sin\left(\frac{n\pi}{2}\right)\right]$

$$\left[\because r=\sqrt{0+4}=2 \quad \theta=\tan^{-1}\left(\frac{2}{0}\right)=\tan^{-1}(\infty)=\frac{\pi}{2}\right].$$

We observe that R.H.S of equation is of the form $2^n\cos\left(\frac{n\pi}{2}\right)$ i.e $r^n\cos n\theta$

We take $a_n^{(p)}$ is in the form

$$a_n^{(p)} = n.2^n\left[A_1\cos\left(\frac{n\pi}{2}\right)+A_2\sin\left(\frac{n\pi}{2}\right)\right].$$

$\left(\because 2^n\cos\left(\frac{n\pi}{2}\right)\right.$ is a root of solution of characteristic equation$\left.\right)$

$$a_{n+1}^{(p)} = (n+1)2^{n+1}\left[A_1\cos(n+1)\frac{\pi}{2}+A_2\sin(n+1)\frac{\pi}{2}\right]$$

$$a_{n+2}^{(p)} = (n+2)2^{n+2}\left[A_1\cos(n+2)\frac{\pi}{2}+A_2\sin(n+2)\frac{\pi}{2}\right]$$

$$=(n+2)2^{n+2}\left[-A_1\cos\left(\frac{n\pi}{2}\right)-A_2\sin\left(\frac{n\pi}{2}\right)\right].$$

Sub. these values in (1), we get

$$(n+2)2^{n+2}\left[-A_1\cos\left(\frac{n\pi}{2}\right)-A_2\sin\left(\frac{n\pi}{2}\right)\right]+4n2^n\left[A_1\cos\left(\frac{n\pi}{2}\right)+A_2\sin\left(\frac{n\pi}{2}\right)\right]$$

$$=2^n\cos\left(\frac{n\pi}{2}\right).$$

Comparing the coefficients of $\cos\left(\frac{n\pi}{2}\right)$ and $\sin\left(\frac{n\pi}{2}\right)$ both sides, we get

$$-8A_1 = 1 \implies A_1 = -\frac{1}{8} \qquad -8A_2 = 0 \implies A_2 = 0.$$

$$\therefore\ a_n^{(p)} = -n\,2^n \frac{1}{8} \cos\left(\frac{n\pi}{2}\right).$$

$\therefore$ The general solution of ① is $a_n = a_n^{(h)} + a_n^{(p)}$.

$$a_n = 2^n\left[A_1 \cos\left(\frac{n\pi}{2}\right) + A_2 \sin\left(\frac{n\pi}{2}\right)\right] - \frac{2^n n}{8} \cos\left(\frac{n\pi}{2}\right).$$

Find the generating functions for the following sequences

(a) 1, 2, 3, 4 . . . . (b) 1, −2, 3, −4, ... (c) 0, 1, 2, 3, ...

(d) 0, 1, −2, 3, −4, ...

Sol: (a) Here $a_0 = 1$ $a_1 = 2$ $a_2 = 3$, $a_3 = 4$ . . . .

A generating function for this sequence is given by.

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$$

$$f(x) = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots$$

$$f(x) = (1-x)^{-2}.$$

(b) Here $a_0 = 1$ $a_1 = -2$ $a_2 = 3$ $a_3 = -4$, ...

A generating function for this sequence is given by.

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$$

$$f(x) = 1 - 2x + 3x^2 - 4x^3 + 5x^4 + \cdots$$

$$f(x) = (1+x)^{-2}.$$

(c) Here $a_0 = 0$ $a_1 = 1$ $a_2 = 2$ $a_3 = 3$, $a_4 = 4$ . . . .

A generating function for this sequence is given by.

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$$

$$= 0 + x + 2x^2 + 3x^3 + 4x^4 + \cdots$$

$$f(x) = x(1 + 2x + 3x^2 + 4x^3 + \cdots).$$

$$f(x) = x(1-x)^{-2}.$$

(d) Here $a_0 = 0$ $a_1 = 1$ $a_2 = -2$ $a_3 = 3$ $a_4 = -4$ . . .

A generating function for this sequence is given by.

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$$

$$= 0 + x - 2x^2 + 3x^3 - 4x^4 + \cdots$$

$$= x(1 - 2x + 3x^2 - 4x^3 + \cdots)$$

$$f(x) = x(1+x)^{-2}.$$

Find the generating functions for the following sequences.

(a) $1^2, 2^2, 3^2, \ldots$ (b) $0^2, 1^2, 2^2, 3^2, \ldots$ (c) $1^3, 2^3, 3^3 \ldots$ (d) $0^3, 1^3, 2^3, 3^3, \ldots$

Sol: (a) The generating function for the sequence $0, 1, 2, 3, \ldots$ is $x(1-x)^{-2}$.

i.e $0 + 1x + 2x^2 + 3x^3 + \cdots = x(1-x)^{-2}$.

Diff. w.r.t "x", we get

$$1 + 2^2 x + 3^2 x^2 + \cdots = \frac{d}{dx}\left\{\frac{x}{(1-x)^{2}}\right\} = \frac{1+x}{(1-x)^3}$$

$\therefore f(x) = \frac{1+x}{(1-x)^3}$ is a generating function for the sequence $1^2, 2^2, 3^2, 4^2, \ldots$

(b) Here $a_0 = 0^2 \quad a_1 = 1^2 \quad a_2 = 2^2 \quad a_3 = 3^2 \ldots$

The generating function for this sequence is given by.

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

$$f(x) = 0 + 1^2 x + 2^2 x^2 + 3^2 x^3 + \cdots$$

$$f(x) = x\left[1^2 + 2^2 x + 3^2 x^2 + 4^2 x^3 + \cdots\right].$$

$$f(x) = x \cdot \frac{(1+x)}{(1-x)^3}$$

$\therefore f(x) = \frac{x(1+x)}{(1-x)^3}$ is a generating function for the sequence $0^2, 1^2, 2^2, 3^2 \ldots$

(c) The generating function for the sequence $0^2, 1^2, 2^2, 3^2 \ldots$ is $\frac{x(1+x)}{(1-x)^3}$.

i.e $0^2 x + 1^2 \cdot x + 2^2 x^2 + 3^2 \cdot x^3 + \cdots = \frac{x(1+x)}{(1-x)^3} = \frac{x^2+x}{(1-x)^3}$.

Diff. w.r.t 'x', both sides, we get:

$$1 + 2^3 \cdot x + 3^3 x^2 + 4^3 x^3 + \cdots = \frac{d}{dx}\left\{\frac{x^2+x}{(1-x)^3}\right\} = \frac{x^2+4x+1}{(1-x)^4}.$$

$\therefore f(x) = \frac{x^2+4x+1}{(1-x)^4}$ is a generating function for the sequence

$1^3, 2^3, 3^3, 4^3, \ldots$

1d) Given that $0^3, 1^3, 2^3, 3^3, \ldots$

$a_0 = 0 \quad a_1 = 1^3 \quad a_2 = 2^3 \quad a_3 = 3^3 \ldots$

A generating function for this sequence is given by

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$$

$$f(x) = 0 + 1^3 x + 2^3 x^2 + 3^3 x^3 + \ldots$$

$$f(x) = x[1^3 + 2^3 x + 3^3 x^2 + \ldots]$$

$$f(x) = x \cdot \frac{x^2+4x+1}{(1-x)^4} \qquad \left[\because 1 + 2^3 x + 3^3 x^2 + \ldots = \frac{x^2+4x+1}{(1-x)^4}\right]$$

This is the generating function for the sequence $0^3, 1^3, 2^3, 3^3 \ldots$

Find the generating function of $\frac{1}{n!}$

Sol:- Let $a_n = \frac{1}{n!}$

$a_0 = 1 \quad a_1 = 1 \quad a_2 = \frac{1}{2!} \quad a_3 = \frac{1}{3!} \ldots$

A generating function for this sequence is given by

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$$

$$f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

$$f(x) = e^x.$$

$\therefore$ A generating function of the sequence is $1, 1, \frac{1}{2!}, \frac{1}{3!} \ldots$ is e^x.

Find the generating function of the sequence $a_n = n^2 + n$.

Sol:- Given that $a_n = n^2 + n$.

The sequence is $0, 1^2+1, 2^2+2, 3^2+3, \ldots$

We can write this sequence as the sum of two sequences

$0, 1^2, 2^2, 3^2, \ldots \qquad 0, 1, 2, 3, \ldots$

Suppose generating function of the sequence u_{n_1} is $g_1(x)$ and the generating function of the sequence u_{n_2} is $g_2(x)$ then the generating function of $u_n = u_{n_1} + u_{n_2}$ is $g_1(x) + g_2(x)$.

$$g_1(x) = \sum_{n=0}^{\infty} u_{n_1} x^n = 0 + x + 2^2 x^2 + 3^2 x^3 + 4^2 x^3 + \cdots$$

$$g_1(x) = \frac{x(1+x)}{(1-x)^3}.$$

The generating function of $0, 1, 2, \ldots$ is

$$g_2(x) = x + 2x^2 + 3x^3 + \cdots = x(1 + 2x + 3x^2 + \cdots)$$

$$g_2(x) = x(1-x)^{-2} = \frac{x}{(1-x)^2}.$$

The generating function of $u_n = u_{n_1} + u_{n_2}$ is $g_1(x) + g_2(x)$

$$\frac{x(1+x)}{(1-x)^3} + \frac{x}{(1-x)^2}$$

$$= \frac{x(1+x) + x(1-x)}{(1-x)^3}$$

$$= \frac{2x}{(1-x)^3}$$

So the generating function of (n^2+n) is $\frac{2x}{(1-x)^3}$.

Find the generating function of $n^2 - 2$

Sol: Let $a_n = n^2 - 2$

The sequence is $0^2-2, 1^2-2, 2^2-2, 3^2-2, \ldots$

We can write this sequence as the sum of two sequences.

$0^2, 1^2, 2^2, 3^2, \ldots, -2, -2, -2, -2, \ldots$

Suppose generating function of the sequence a_{n_1} is $g_1(x)$ and the generating function of the sequence a_{n_2} is $g_2(x)$ then the generating function of $a_n = a_{n_1} + a_{n_2}$ is $g_1(x) + g_2(x)$

The generating function of $0^2, 1^2, 2^2, \ldots$ (or) n^2 is $g_1(x) = \frac{x(1+x)}{(1-x)^3}$.

The generating function of $1, 1, 1, \ldots$ is $g_2(x) = \frac{1}{1-x}$.

So the generating function of n^2-2 is $g_1(x)-2g_2(x)$

$$= \frac{x(1+x)}{(1-x)^3} - \frac{2}{(1-x)}$$

$$= \frac{x(1+x)-2(1-x)^2}{(1-x)^3}$$

$$= \frac{x+x^2-2-2x^2+4x}{(1-x)^3}$$

$$= \frac{5x-x^2-2}{(1-x)^3}.$$

$\therefore$ The generating function of n^2-2 is $\frac{5x-x^2-2}{(1-x)^3}$

Find the generating function of $(n-1)^2$.

Sol:- Let $u_n = n^2+1-2n$.

The generating function of u_n is the sum of three sequences $0^2, 1^2, 2^2, \ldots$, $1, 1, 1, \ldots$ and $0, -2, -4, -6, \ldots$

suppose generating function of the sequence u_{n_1} is $g_1(x)$ and the generating function of the sequence u_{n_2} is $g_2(x)$ and the generating function of the sequence u_{n_3} is $g_3(x)$.

Then the generating function of $u_n = u_{n_1}+u_{n_2}+u_{n_3} = g_1(x)+g_2(x)+g_3(x)$.

The generating function of n^2 is $g_1(x) = \frac{x(1+x)}{(1-x)^3}$.

The generating function of n is $g_2(x) = \frac{x}{(1-x)^2}$.

The generating function of 1 is $g_3(x) = \frac{1}{1-x}$.

The generating function of $(n-1)^2$ is $g_1(x)-2g_2(x)+g_3(x)$

$$= \frac{x(1+x)}{(1-x)^3} - \frac{2x}{(1-x)^3} + \frac{1}{1-x}$$

Find the generating function of a^n (a is a constant)

Sol: Let $u_n = a^n$.

The generating function of u_n is

$$f(x) = \sum_{n=0}^{\infty} u_n x^n$$

$$= \sum_{n=0}^{\infty} a^n x^n = \sum_{n=0}^{\infty} (ax)^n$$

$$= 1 + (ax) + (ax)^2 + (ax)^3 + \cdots$$

$$= (1-ax)^{-1}$$

$$f(x) = \frac{1}{1-ax}.$$

The generating function of a^n (a is constant) is $\frac{1}{1-ax}$.

Find the generating function of na^n ("a" is a constant)

Sol: Let $u_n = na^n$.

The generating function of na^n is

$$f(x) = \sum_{n=0}^{\infty} u_n x^n = \sum_{n=0}^{\infty} n a^n x^n$$

$$= 0 + ax + 2a^2x^2 + 3a^3x^3 + 4a^4x^4 + \cdots$$

$$= x\left[a + 2a^2x + 3a^3x^2 + 4a^4x^3 + \cdots\right].$$

$$= x \cdot \frac{d}{dx}\left[ax + a^2x^2 + a^3x^3 + a^4x^4 + \cdots\right]$$

$$= x \cdot \frac{d}{dx}\left[(1-ax)^{-1} - 1\right]$$

$$= x \cdot \left[\frac{d}{dx}(1-ax)^{-1} - \frac{d}{dx}(1)\right]$$

$$= x \cdot \frac{a}{(1-ax)^2} - 0 = \frac{ax}{(1-ax)^2}.$$

$\therefore$ The generating function of na^n is $\frac{ax}{(1-ax)^2}$.

Find the generating function of $\sin(\frac{n\pi}{2})$.

Sol:- Let $a_n = \sin(\frac{n\pi}{2})$

The generating function of a_n is given by

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \sin(\tfrac{n\pi}{2}) \cdot x^n$$

$$= 0 + \sin(\tfrac{\pi}{2}) x + \sin(\pi) \cdot x^2 + \sin(\tfrac{3\pi}{2}) x^3 + \cdots$$

$$= x - x^3 + x^5 - x^7 + x^9 + \cdots$$

[When n is even $\sin(\frac{n\pi}{2}) = 0$, when n is $4n-1$, $\sin(\frac{n\pi}{2}) = -1$.

when n is $4n+1$ $\sin(\frac{n\pi}{2}) = 1$ when $n=1$ $\sin(\frac{n\pi}{2}) = 1$]

$$= x[1 - x^2 + x^4 - x^6 + x^8 \cdots]$$

$$= x (1 + x^2)^{-1}$$

$$= \frac{x}{1+x^2}.$$

Find the generating function of $\cos(\frac{n\pi}{2})$.

Sol:- Let $a_n = \cos(\frac{n\pi}{2})$.

The generating function of a_n is given by. $f(x) = \sum_{n=0}^{\infty} a_n x^n$.

$$f(x) = \sum_{n=0}^{\infty} \cos(\tfrac{n\pi}{2}) x^n$$

$$= \cos 0 . x^0 + \cos(\tfrac{\pi}{2}) \cdot x + \cos(\pi) x^2 + \cos(\tfrac{3\pi}{2}) \cdot x^3 + \cdots$$

$$= 1 - x^2 + x^4 - x^6 + x^8 + \cdots$$

[$\cos(\frac{n\pi}{2}) = 0$ when n is odd., $\cos(\frac{n\pi}{2}) = -1$ when n is $4m-2$.

$\cos(\frac{n\pi}{2}) = 1$ when n is $4m$.]..

$$f(x) = (1+x^2)^{-1} = \frac{1}{1+x^2}$$

$\therefore$ The generating function of $\cos(\frac{n\pi}{2})$ is $\frac{1}{1+x^2}$.

Find the generating function of $4^n + 2n$.

Sol: Given that $a_n = 4^n + 2n$, The sequence is $1+0,\ 4+2,\ 4^2+4,\ 4^3+6, \dots$

We can write this sequence as the sum of two sequences.

$1, 4, 4^2, 4^3, \dots \quad , 0, 2, 4, 6, \dots$

Suppose generating function of the sequence u_{n_1} is $g_1(x)$ and the generating function of the sequence u_{n_2} is $g_2(x)$ then the generating function of $u_n = u_{n_1} + u_{n_2}$ is $g_1(x) + g_2(x)$

$$g_1(x) = 1 + 4x + 4^2x^2 + 4^3x^3 + \cdots$$

$$g_1(x) = (1-4x)^{-1} = \frac{1}{1-4x}.$$

$$g_2(x) = 0 + 2x + 4x^2 + 6x^3 + \cdots$$

$$g_2(x) = 2x[1 + 2x + 3x^2 + \cdots] = 2x(1-x)^{-2} = \frac{2x}{(1-x)^2}.$$

$\therefore$ The generating function of $4^n + 2n$ is

$$f(x) = g_1(x) + g_2(x)$$

$$f(x) = \frac{1}{1-4x} + \frac{2x}{(1-x)^2}$$

$$= \frac{(1-x)^2 + 2x(1-4x)}{(1-4x)(1-x)^2}$$

$$= \frac{1+x^2-2x+2x-8x^2}{(1-4x)(1-x)^2}$$

$$f(x) = \frac{1-7x^2}{(1-4x)(1-x)^2}$$

$\therefore$ This is the generating function of $4^n + 2n$.

Solving Recurrence Relations by Generating Functions :—

Method of generating functions for first order recurrence relations :-

Suppose the recurrence relation to be solved is of the form.

$$a_n = ca_{n-1} + F(n) \quad \text{for } n \geq 1.$$

or equivalently $a_{n+1} = ca_n + \phi(n)$ for $n \geq 0$ —— ①.

Where c is a constant and $\phi(n) = F(n+1)$. is a given function of n.

Let us multiply bothsides of the relation ① by x^{n+1} :

$$a_{n+1} x^{n+1} = ca_n x^{n+1} + \phi(n) x^{n+1}.$$

Taking the sum of all the relations got from this relation for $n = 0, 1, 2 \ldots$

we obtain $\quad \sum_{n=0}^{\infty} a_{n+1} x^{n+1} = c \sum_{n=0}^{\infty} a_n x^{n+1} + \sum_{n=0}^{\infty} \phi(n) x^{n+1}$

This can be rewritten as.

$$\sum_{n=1}^{\infty} a_n x^n - cx \sum_{n=0}^{\infty} a_n x^n = x \sum_{n=0}^{\infty} \phi(n) x^n.$$

$$\left[\sum_{n=0}^{\infty} a_n x^n - a_0\right] - cx \sum_{n=0}^{\infty} a_n x^n = x \sum_{n=0}^{\infty} \phi(n) x^n.$$

$$(1-cx) \sum_{n=0}^{\infty} a_n x^n = a_0 + x \sum_{n=0}^{\infty} \phi(n) x^n \quad \text{—— ②}.$$

If $f(x)$ is a generating function for the sequence

$\langle a_r \rangle = \langle a_n \rangle = a_0, a_1, a_2, \ldots$ then we have $f(x) = \sum_{n=0}^{\infty} a_n x^n$ —— ③.

Here, we say that $f(x)$ is a generating function for the recurrence relation ①.

In the recurrence relation ①, $\phi(n)$ is a known function of n.

If we set $g(x) = \sum_{n=0}^{\infty} \phi(n) x^n$ —— ④. then $g(x)$ is a generating function for the sequence $\langle \phi(n) \rangle = \phi(0), \phi(1), \phi(2) \ldots$ —— ⑤.

Using ③ and ④ in ②, we obtain $(1-cx) f(x) = a_0 + x g(x)$.

$$f(x) = \frac{a_0 + x g(x)}{1-cx} \quad \text{—— ⑥}.$$

When a generating function $g(x)$ of the sequence (5) is known, this expression determines the generating function $f(x)$ for the recurrence relation ①. This function is unique for a specified a_0.

since a_n is the coefficient of x^n in the expansion of $f(x)$, as is evident from ③, the coefficient of x^n in the RHS of ⑥ determines a_n. Thus the relation ① is solved.

Find the generating function for the recurrence relation $a_{n+1} - a_n = 3^n, n \geq 0$ and $a_0 = 1$. Hence solve the relation.

Sol: Given that the recurrence relation $a_{n+1} = a_n + 3^n, n \geq 0$ —①

Compare the given recurrence relation with $a_{n+1} = ca_n + \phi(n)$ —② for $n \geq 0$

Here $c = 1$ $\phi(n) = 3^n$

$\therefore$ The generating function for the relation is given by

$$f(x) = \frac{a_0 + x\,g(x)}{1 - cx} \quad \text{where } g(x) = \sum_{n=0}^{\infty} \phi(n)\, x^n.$$

$$f(x) = \frac{a_0 + x\,g(x)}{1 - x} \quad —③$$

$$g(x) = \sum_{n=0}^{\infty} \phi(n)\, x^n = \sum_{n=0}^{\infty} 3^n \cdot x^n = \sum_{n=0}^{\infty} (3x)^n$$

$$g(x) = 1 + (3x)^1 + (3x)^2 + (3x)^3 + \cdots$$

$$g(x) = (1-3x)^{-1}$$

Given that $a_0 = 1$

From ③, $$f(x) = \frac{1 + x(1-3x)^{-1}}{(1-x)}$$

$$f(x) = \frac{(1-3x) + x}{(1-3x)(1-x)} = \frac{1-2x}{(1-3x)(1-x)} \quad —④$$

$$\frac{1-2x}{(1-3x)(1-x)} = \frac{A}{1-3x} + \frac{B}{1-x}.$$

$$(1-2x) = A(1-3x) + B(1-x).$$

Equating the corresponding co efficients in this, we get

$$A+B=1 \qquad -2 = -3A-B.$$

Solving these, we get $A=B=\frac{1}{2}$.

$$\frac{1-2x}{(1-x)(1-3x)} = \frac{1}{2}\left(\frac{1}{1-x} + \frac{1}{1-3x}\right)$$

$$\therefore \quad f(x) = \frac{1}{2}\left(\frac{1}{1-x} + \frac{1}{1-3x}\right)$$

$$f(x) = \frac{1}{2}\left[(1-x)^{-1} + (1-3x)^{-1}\right]$$

$$f(x) = \frac{1}{2}\left[(1+x+x^2+\cdots) + (1+(3x)+(3x)^2+\cdots)\right]$$

$$f(x) = \frac{1}{2}\left[\sum_{n=0}^{\infty} x^n + \sum_{n=0}^{\infty}(3x)^n\right]$$

$$f(x) = \frac{1}{2}\left[\sum_{n=0}^{\infty} x^n + (3x)^n\right] = \frac{1}{2}\sum_{n=0}^{\infty}(1+3^n)x^n.$$

$$f(x) = \frac{1}{2}\sum_{n=0}^{\infty}(1+3^n)x^n.$$

Since $f(x) = \sum_{n=0}^{\infty} a_n x^n$.

We find that $a_n = \frac{1}{2}(1+3^n)$.

Find a generating function for the recurrence relation $a_{n+1} - a_n = n^2$, $n \geq 0$. and $a_0 = 1$. Hence solve it.

Sol:- Given that $a_{n+1} = a_n + n^2$, $n \geq 0$.

The given relation is of the form $a_{n+1} = ca_n + \phi(n)$.

where $c=1$ and $\phi(n) = n^2$.

$\therefore$ A generating function for the relation is given by

$$f(x) = \frac{a_0 + x\,g(x)}{1-cx}. \qquad \text{where } g(x) = \sum_{n=0}^{\infty}\phi(n)x^n.$$

$$f(x) = \frac{1+x\,g(x)}{1-x}.$$

$$g(x) = \sum_{n=0}^{\infty} n^2 x^n.$$

This means that $g(x)$ is a generating function for the sequence.

$$\langle n^2 \rangle = 0^2, 1^2, 2^2, 3^2, \ldots$$

$$\ldots \quad g(x) = \frac{x(1+x)}{(1-x)^3}$$

$$f(x) = \frac{1}{1-x} \left\{ 1 + \frac{x^2(1+x)}{(1-x)^3} \right\}$$

$$= \frac{1}{(1-x)} \left\{ \frac{(1-x)^3 + x^2(1+x)}{(1-x)^3} \right\}$$

$$f(x) = \frac{(1-x)^3 + x^2(1+x)}{(1-x)^4} = \frac{1-3x+4x^2}{(1-x)^4}.$$

This is the generating function for the given relation.

$$f(x) = (1-3x+4x^2)(1-x)^{-4}.$$

WKT If n is a positive integer $(1-x)^{-n} = \sum_{r=0}^{\infty} \binom{n+r-1}{r} x^r.$

$$f(x) = (1-3x+4x^2) \sum_{r=0}^{\infty} \binom{4+r-1}{r} x^r$$

$$= (1-3x+4x^2) \sum_{r=0}^{\infty} \binom{3+r}{r} x^r$$

$$= \sum_{r=0}^{\infty} \binom{3+r}{r} x^r - 3 \sum_{r=0}^{\infty} \binom{3+r}{r} x^{r+1} + 4 \sum_{r=0}^{\infty} \binom{3+r}{r} x^{r+2} \quad \text{--- (i)}$$

Since $f(x) = \sum_{n=0}^{\infty} a_n x^n$

We find that a_n = co efficient of x^n in the R.H.S. of (i).

$$= \binom{3+n}{n} - 3\binom{3+n-1}{n-1} + 4\binom{3+n-2}{n-2}$$

$$a_n = {}^{(n+3)}C_n - 3\,{}^{(n+2)}C_{n-1} + 4\,{}^{(n+1)}C_{(n-2)}.$$

$$a_n = \frac{(n+3)!}{n!\,3!} - 3\frac{(n+2)!}{(n-1)!\,3!} + 4\frac{(n+1)!}{(n-2)!\,3!}$$

$$a_n = \frac{(n+3)(n+2)(n+1)\cancel{n!}}{\cancel{n!}\,3!} - 3\,\frac{(n+2)(n+1)n\,\cancel{(n-1)!}}{\cancel{(n-1)!}\,3!} + 4\,\frac{(n+1)n(n-1)\cancel{(n-2)!}}{\cancel{(n-2)!}\,3!}$$

$$= \frac{(n+1)}{6}\left[(n^2+5n+6) - 3(n^2+2n) + 4(n^2-n)\right]$$

$$a_n = \frac{(n+1)}{6}(2n^2-5n+6)$$

$\therefore$ This is the required solution.

Method of generating functions for second order Recurrence Relations :-

Suppose the recurrence relation to be solved is of the form.

$$a_n + A a_{n-1} + B a_{n-2} = F(n) \quad \text{for } n \geq 2.$$

or equivalently $a_{n+2} + A a_{n+1} + B a_n = \phi(n)$ for $n \geq 0$. —— ①.

Where A and B are known constants and $\phi(n) = F(n+2)$ is a specified function.

Let us multiply both sides of the relation ① by x^{n+2} and take the sum of all the resulting relations the corrospond to $n = 0, 1, 2, 3, \ldots$ Then, we obtain.

$$\sum_{n=0}^{\infty} a_{n+2}\, x^{n+2} + A \sum_{n=0}^{\infty} a_{n+1}\, x^{n+2} + B \sum_{n=0}^{\infty} a_n\, x^{n+2} = \sum_{n=0}^{\infty} \phi(n)\, x^{n+2}.$$

This may be rewritten as

$$\sum_{n=2}^{\infty} a_n x^n + A x \sum_{n=1}^{\infty} a_n x^n + B x^2 \sum_{n=0}^{\infty} a_n x^n = x^2 \sum_{n=0}^{\infty} \phi(n)\, x^n.$$

$$\left(\sum_{n=0}^{\infty} a_n x^n - a_0 - a_1 x\right) + A x\left(\sum_{n=0}^{\infty} a_n x^n - a_0\right) + B x^2 \sum_{n=0}^{\infty} a_n x^n = x^2 \sum_{n=0}^{\infty} \phi(n)\, x^n.$$

$$(1 + Ax + Bx^2) \sum_{n=0}^{\infty} a_n x^n = a_0 + (a_1 + a_0 A)x + x^2 \sum_{n=0}^{\infty} \phi(n)\, x^n \quad \text{—— ②}.$$

Let $f(x)$ be a generating function for the sequence $\langle a_r \rangle$ for which ① is the recurrence relation. Then, we have.

$$f(x) = \sum_{n=0}^{\infty} a_n x^n. \quad \text{—— ③}.$$

Here we say that $f(x)$ is a generating function for the relation ①.

In the relation ①, $\phi(n)$ is a known function of n.

If we set $g(x) = \sum_{n=0}^{\infty} \phi(n)\, x^n$ —— ④.

then $g(x)$ is a generating function for the sequence $\langle \phi(n) \rangle = \phi(0), \phi(1), \ldots$ —— ⑤.

sub. ③ and ④ in ②, we get

$$(1 + Ax + Bx^2) f(x) = a_0 + (a_1 + a_0 A)x + x^2 g(x).$$

$$f(x) = \frac{a_0 + (a_1 + a_0 A)x + x^2 g(x)}{1 + Ax + Bx^2} \quad \text{——— ⑥.}$$

When the generating function $g(x)$ of the sequence ⑤ is known, this expression determines the generating function $f(x)$ for the recurrence relation ①. This function is unique for specified a_0 and a_1. Since a_n is the coefficient of x^n in the expansion of $f(x)$, as is evident from ③, the coefficient of x^n in the RHS of ⑥ determines a_n. Thus the relation ① is solved.

Note:- If the relation ① is homogeneous, that is if $\phi(n) = 0$ then $g(x) = 0$ and expression ⑥ becomes $f(x) = \dfrac{a_0 + (a_1 + a_0 A)x}{1 + Ax + Bx^2}$ ——— ⑦.

Solve the recurrence relation $a_{n+2} - 2a_{n+1} + a_n = 2^n$, $n \geq 0$. and $a_0 = 1$. $a_1 = 2$ by the method of generating function. (5)

Sol:- Given that the recurrence relation $a_{n+2} - 2a_{n+1} + a_n = 2^n$, $n \geq 0$. (1)

Compare the relation (1) with $a_{n+2} + A a_{n+1} + B a_n = \phi(n)$

$$A = -2 \quad B = 1 \quad \phi(n) = 2^n$$

$\therefore$ The generating function for the given relation is.

$$f(x) = \frac{a_0 + (a_1 + a_0 A)x + x^2 g(x)}{1 + Ax + Bx^2}$$

$$f(x) = \frac{a_0 + (a_1 - 2a_0)x + x^2 g(x)}{1 - 2x + x^2}$$

We have $a_0 = 1 \quad a_1 = 2$

$$f(x) = \frac{1 + x^2 g(x)}{(1-x)^2}.$$

Here $g(x) = \sum_{n=0}^{\infty} \phi(n) \cdot x^n$

$$g(x) = \sum_{n=0}^{\infty} 2^n \cdot x^n = 1 + (2x)^1 + (2x)^2 + \cdots$$

$$g(x) = (1-2x)^{-1}.$$

$$f(x) = \frac{1}{(1-x)^2}\left[1 + x^2 \frac{1}{(1-2x)}\right]$$

$$= \frac{1}{(1-x)^2}\left[\frac{1 - 2x + x^2}{(1-2x)}\right]$$

$$f(x) = \frac{1}{1-2x}.$$

This is the generating function for the given relation

$$f(x) = \frac{1}{1-2x} = (1-2x)^{-1} = \sum_{n=0}^{\infty} (2x)^n = \sum_{n=0}^{\infty} 2^n \cdot x^n.$$

since $f(x) = \sum_{n=0}^{\infty} a_n x^n$.

$\therefore$ $a_n = 2^n$. This is the required solution.

Using generating function, solve $a_{n+2} - 4a_{n+1} + 3a_n = 0$., given $a_0 = 2, a_1 = 4$.

Sol: Given that the relation $a_{n+2} - 4a_{n+1} + 3a_n = 0$ —— ① . for $n \geq 0$.

Compare the relation ① with. $a_{n+2} + Aa_{n+1} + Ba_n = \phi(n)$ for $n \geq 0$.

$$A = -4 \quad B = \quad \phi(n) = 0.$$

∴ A generating function for the relation is .

$$f(x) = \frac{a_0 + (a_1 + a_0 A)x}{1 + Ax + Bx^2}.$$

$$f(x) = \frac{2 + (4 + 2(-4))x}{1 - 4x + 3x^2} = 2 - 4x.$$

$$f(x) = \frac{2(1-2x)}{(1-3x)(1-x)} = \frac{A}{1-3x} + \frac{B}{1-x}.$$

$$\cancel{f(x) = A}$$

$$\frac{2(1-2x)}{(1-3x)(1-x)} = \frac{A(1-x) + B(1-3x)}{(1-3x)(1-x)}$$

$$2(1-2x) = A(1-x) + B(1-3x)$$

Put $x = 1$, Then $B = 1$.

Put $x = \frac{1}{3}$ Then $A = 1$.

$$\therefore f(x) = \frac{1}{1-3x} + \frac{1}{1-x} = (1-3x)^{-1} + (1-x)^{-1}.$$

$$f(x) = [1 + (3x) + (3x)^2 + (3x)^3 + \cdots] + [1 + x + x^2 + \cdots]$$

$$= \sum_{n=0}^{\infty} (3x)^n + \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} (3^n x^n + x^n) = \sum_{n=0}^{\infty} (1+3^n) x^n.$$

$$f(x) = \sum_{n=0}^{\infty} (1+3^n) x^n.$$

since $f(x) = \sum_{n=0}^{\infty} a_n x^n$.

$$a_n = 1 + 3^n.$$

This is the solution of the given relation.

Find the generating function for the Fibonacci sequence $<F_n>$ and hence obtain an expression for F_n. (53)

Sol:- We know the Febonacci sequence is defined through the recurrence relation is: $F_{n+2} = F_{n+1} + F_n$ for $n \geq 0$ with $F_0 = 0$, $F_1 = 1$.

This relation is homogeneous and the co efficients of F_{n+1} and F_n are constants.

Compare the given relation with. $a_{n+2} + Aa_{n+1} + Ba_n = \phi(n)$ for $n \geq 0$.

$$A = -1 \quad B = -1 \quad \phi(n) = 0$$

The generating function for F_n is

$$f(x) = \frac{F_0 + (F_1 + F_0 A)x}{1 + Ax + Bx^2} = \frac{0 + (1 + 0.(-1))x}{1 - x - x^2}.$$

$$f(x) = \frac{-x}{x^2 + x - 1}.$$

We note that the roots of $x^2 + x - 1 = 0$ are.

$$\alpha, \beta = \frac{-1 \pm \sqrt{1+4}}{2} = \frac{1}{2}(-1 \pm \sqrt{5})$$

So that $x^2 + x - 1 = (x - \alpha)(x - \beta)$.

$$\text{Let } f(x) = \frac{-x}{x^2 + x - 1} = \frac{A}{x - \alpha} + \frac{B}{x - \beta}.$$

$$-x = A(x - \beta) + B(x - \alpha)$$

Put $x = \alpha$, $A = \dfrac{1}{\beta - \alpha}$ Put $x = \beta$ $B = \dfrac{\beta}{\alpha - \beta}$.

With A and B determined as above, we have.

$$f(x) = \frac{A}{x - \alpha} + \frac{B}{x - \beta}.$$

$$f(x) = \frac{-A}{\alpha(1-\frac{x}{\alpha})} - B\frac{1}{\beta(1-\frac{x}{\beta})}$$

$$= -\frac{A}{\alpha}\left(1-\frac{x}{\alpha}\right)^{-1} - \frac{B}{\beta}\left(1-\frac{x}{\beta}\right)^{-1}$$

$$= -\frac{A}{\alpha}\left[1+\frac{x}{\alpha}+\left(\frac{x}{\alpha}\right)^2+\cdots\right] - \frac{B}{\beta}\left[1+\left(\frac{x}{\beta}\right)+\left(\frac{x}{\beta}\right)^2+\cdots\right]$$

$$= -\frac{A}{\alpha}\sum_{n=0}^{\infty}\left(\frac{x}{\alpha}\right)^n - \frac{B}{\beta}\sum_{n=0}^{\infty}\left(\frac{x}{\beta}\right)^n$$

$$= -\sum_{n=0}^{\infty}\left(\frac{A}{\alpha^{n+1}} + \frac{B}{\beta^{n+1}}\right)x^n.$$

Since $f(x)$ is the generating function for F_n, the co eff. of x^n in $f(x)$ is equal to F_n.

$$F_n = -\left(\frac{A}{\alpha^{n+1}} + \frac{B}{\beta^{n+1}}\right) = \frac{-1}{(\alpha\beta)^n}\left[\frac{A\beta^n}{\alpha} + \frac{B\alpha^n}{\beta}\right]$$

$$F_n = -\frac{1}{(-1)^n}\left[\frac{1}{(\beta-\alpha)}\beta^n + \frac{1}{(\alpha-\beta)}\cdot\alpha^n\right].$$

$$= \frac{(-1)^n}{\sqrt{5}}(\beta^n - \alpha^n) = \frac{1}{\sqrt{5}}\left[(-\beta)^n - (-\alpha)^n\right]$$

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

$\because \beta = \frac{-1-\sqrt{5}}{2}$

$\alpha = \frac{-1+\sqrt{5}}{2}$

$\beta - \alpha = -\sqrt{5}$

$\alpha\beta = -1$

Which is the required solution of given recurrence relation.

A Counting Technique:—

Suppose we wish to determine the number of integer solutions of the equation $x_1 + x_2 + x_3 + \cdots + x_n = r$ where $n \geq r \geq 0$. under the constraints that x_1 can take the integer values $P_{11}, P_{12}, P_{13} \cdots$

x_2 can take the integer values $P_{21}, P_{22}, P_{23}, \cdots$

$- - - - - - - - - - - -$

x_n can take the integer values $P_{n1}\ P_{n2}\ P_{n3} \cdots$

To solve this problem we first define the functions $f_1(x), f_2(x) \ldots f_n(x)$ as follows.

$$f_1(x) = x^{P_{11}} + x^{P_{12}} + x^{P_{13}} + \cdots$$

$$f_2(x) = x^{P_{21}} + x^{P_{22}} + x^{P_{23}} + \cdots$$

$$\cdots$$

$$f_n(x) = x^{P_{n1}} + x^{P_{n2}} + x^{P_{n3}} + \cdots$$

We then consider the function $f(x)$ defined by.

$$f(x) = f_1(x).f_2(x).f_3(x) \cdots f_n(x).$$

and determine the co efficient of x^r in this function. This co efficient happens to be equal to the number of solutions that we desired to find. The function $f(x)$ is called the generating fun. for the problem.

(1) Using generating function, find the number of (i) non negative and (ii) positive integer solutions of the equation. $x_1 + x_2 + x_3 + x_4 = 25$.

sol:- (i) In the case of non negative integer solutions x_i's can take the values $0, 1, 2, 3, \ldots$

Let us take $f_1(x) = x^0 + x^1 + x^2 + x^3 + \cdots$, $f_3(x) = x^0 + x^1 + x^2 + \cdots$

$f_2(x) = x^0 + x^1 + x^2 + x^3 + \cdots$ $\quad f_4(x) = x^0 + x^1 + x^2 + \cdots$

$\therefore$ The generating function is

$$f(x) = f_1(x).f_2(x).f_3(x).f_4(x).$$

$$f(x) = (1 + x + x^2 + x^3 + x^4 + \cdots)^4.$$

$$f(x) = [(1-x)^{-1}]^4 = (1-x)^{-4} \quad \text{——— (1)}$$

We know that if n is a positive integer, $(1-x)^{-n} = \sum_{r=0}^{\infty} \binom{n+r-1}{r} x^r$.

The co efficient of x^{25} in (1) is $\binom{4+25-1}{25} = \binom{28}{25} = \frac{28!}{25!\,3!} = 3276$.

$\therefore$ The given equation has 3276 non -ve integer solutions.

(ii) Given that $x_1 + x_2 + x_3 + x_4 = 25$.

In the case of positive integer solutions x_i's can take the values 1, 2, 3, 4 . . . .

Let us take $f_1(x) = x + x^2 + x^3 + \cdots$ $\quad f_2(x) = x + x^2 + x^3 + \cdots$

$f_3(x) = x + x^2 + x^3 + \cdots$ $\quad f_4(x) = x + x^2 + x^3 + \cdots$

$\therefore$ The generating function is

$$f(x) = f_1(x)\, f_2(x) \cdot f_3(x)\, f_4(x)$$

$$f(x) = (x + x^2 + x^3 + \cdots)^4$$

$$= [x(1 + x + x^2 + \cdots)]^4$$

$$f(x) = x^4[(1-x)^{-1}]^4 = x^4 (1-x)^{-4} \quad \text{——— (1)}$$

We know that If n is a positive integer, $(1-x)^{-n} = \sum_{r=0}^{\infty} \binom{n+r-1}{r} x^r$.

The co eff. of x^{25} in (1) is $\binom{4+21-1}{21} = \binom{24}{21} = \frac{24!}{21!\,3!} = 2024$.

$\therefore$ The given equation has 2024 positive integer solutions.

Find the generating function for the number of integer solutions to the equation $x_1+x_2+x_3+x_4=20$. Where $-3 \le x_1$, $-3 \le x_2$ $-5 \le x_3 \le 5$ and $0 \le x_4$. Hence find the number of such solutions. (39)

Sol:- Given that $x_1+x_2+x_3+x_4=20$ — ①

$-3 \le x_1 \quad -3 \le x_2 \quad -5 \le x_3 \le 5 \quad 0 \le x_4$

Let us set $y_1 = x_1+3 \quad y_2 = x_2+3 \quad y_3 = x_3+5 \quad y_4 = x_4$ — ②.

Then $y_1 \ge 0 \quad y_2 \ge 0 \quad 0 \le y_3 \le 10., \quad y_4 \ge 0.$ — ③.

Sub ② in ①, we get

$$(y_1-3)+(y_2-3)+(y_3-5)+y_4 = 20$$

$$y_1+y_2+y_3+y_4 = 31. \quad \text{— ④}$$

Thus the number integer solutions of the given equation under the given constraints is equal to the number of integer solutions of equation ④ under the constraints ③.

Let us take
$$f_1(x) = x^0+x^1+x^2+\dots = (1-x)^{-1}$$
$$f_2(x) = x^0+x^1+x^2+\dots = (1-x)^{-1}$$
$$f_3(x) = x^0+x^1+x^2+\dots+x^{10}$$
$$f_4(x) = x^0+x^1+x^2+\dots = (1-x)^{-1}$$

∴ The generating function ~~for~~ is
$$f(x) = f_1(x)f_2(x)f_3(x)f_4(x) = (1+x+x^2+\dots+x^{10})(1-x)^{-3}$$

We know that if n is a +ve integer $(1-x)^{-n} = \sum_{r=0}^{\infty}\binom{n+r-1}{r}x^r$.

$$f(x) = (1+x+x^2+\dots+x^{10})\sum_{r=0}^{\infty}\binom{3+r-1}{r}x^r.$$

$$f(x) = (1+x+x^2+\dots+x^{10})\sum_{r=0}^{\infty}\binom{2+r}{r}x^r. \quad \text{— ⑤}$$

The co efficient of x^{31} in ⑤ is

$$\binom{33}{31}+\binom{32}{30}+\binom{31}{29}+\dots+\binom{23}{21}.$$

This is the required number of solutions.

In how many ways can 12 oranges be distributed among three children A, B, C so that A gets at least four, B and C get at least two but "C" gets at least two but C gets no more than five?

Sol:- Let x_1 be the number of oranges which A can get, x_2 be the number of oranges which B can get. and x_3 be the number of oranges which C can get.

$$x_1 + x_2 + x_3 = \text{Total no. of oranges} = 12 \quad \text{— (1)}.$$

From the given constraints we note that $x_1 \geq 4$, $x_2 \geq 2$, $2 \leq x_3 \leq 5$ — (2).

The required number is equal to the number of integer solutions of the equation (1) under the constraints (2)

Let us take $f_1(x) = x^4 + x^5 + x^6 + \cdots$

$f_2(x) = x^2 + x^3 + x^4 + \cdots$

$f_3(x) = x^2 + x^3 + x^4 + x^5$.

∴ The generating function is

$$f(x) = f_1(x) f_2(x) f_3(x)$$

$$= (x^4 + x^5 + x^6 + \cdots)(x^2 + x^3 + x^4 + \cdots)(x^2 + x^3 + x^4 + x^5)$$

$$= x^4(1 + x + x^2 + \cdots) x^2 (1 + x + x^2 + \cdots) x^2 (1 + x + x^2 + x^3)$$

$$= x^8 (1-x)^{-1} (1-x)^{-1} (1 + x + x^2 + x^3).$$

$$= x^8 (1 + x + x^2 + x^3)(1-x)^{-2}.$$

$$= (x^8 + x^9 + x^{10} + x^{11})(1-x)^{-2} \quad \text{— (3)}$$

We know that If n is a positive integer, $(1-x)^{-n} = \sum_{n=0}^{\infty} \binom{n+r-1}{r} x^r$

The co efficient of x^{12} in (3) is.

$$= (x^8 + x^9 + x^{10} + x^{11}) \left(\sum_{r=0}^{\infty} \binom{2+r-1}{r} x^r \right.$$

$$= (x^8 + x^9 + x^{10} + x^{11}) \sum_{r=0}^{\infty} \binom{1+r}{r} x^r.$$

The co efficient of x^{12} is.

$$\binom{4+1}{4} + \binom{3+1}{3} + \binom{2+1}{2} + \binom{1+1}{1} = 5 + 4 + 3 + 2 = 14.$$

∴ There are 14 ways of making the distribution

In $(1+x^5+x^9)^{10}$ find (a) the co efficient of x^{23} (b) the co efficient of x^{32}.

sol:- Given that $(1+x^5+x^9)^{10}$.

(i) To find the co efficient of x^{23}.

i.e $e_1+e_2+e_3+\cdots+e_{10}=23$.

$e_i=0,5,9$.

The co efficient of x^{23} can be formed with $e_i=0,5,9$.

when we take two 9's, one 5 and seven 0's.

Hence the co efficient of x^{23} in $(1+x^5+x^9)^{10}$ is $\dfrac{10!}{2!\,1!\,7!}$

(ii) to find the co efficient of x^{32}.

i.e $e_1+e_2+e_3+\cdots+e_{10}=32$.

$e_i=0,5,9$.

The co efficient of x^{32} can be formed with $e_i=0,5,9$.

when we take three 9's, one 5 and six 0's

Hence the co efficient of x^{32} in $(1+x^5+x^9)^{10}$ is $\dfrac{10!}{3!\,1!\,6!}$.

→ Determine the co efficient of x^5 in $(a+bx+cx^2)^{10}$

sol:- Given that $(a+bx+cx^2)^{10}$.

To find the co efficient of x^5.

i.e $e_1+e_2+e_3+\cdots+e_{10}=5$

$e_i=0,1,2$ $\quad$ e_i takes the values 0, 1, 2.

The co efficient of x^5 can be formed with $e_i=0,1,2$.

(i) when we take two 2's, one 1 and seven 0's i.e $\dfrac{10!}{2!\,1!\,7!}a^7c^2b$.

(ii) when we take three 1's, one 2 and six 0's i.e $\dfrac{10!}{3!\,1!\,6!}b^3ca^6$

(iii) when we take five 0's, five 1's i.e $\dfrac{10!}{5!\,5!}a^5b^5$.

Hence the co efficient of x^5 in $(a+bx+cx^2)^{10}$ is

$$\frac{10!}{7!\,1!\,2!}a^7bc^2+\frac{10!}{6!\,3!\,1!}a^6b^3c+\frac{10!}{5!\,5!}a^5b^5.$$

Find the co efficient of x^{14} in $(1+x+x^2+x^3)^{10}$.

Sol:- Given that $(1+x+x^2+x^3)^{10}$.

To find the co efficient of x^{14}

i.e $e_1+e_2+e_3 \cdots + e_{10} = 14$.

$e_i = 0, 1, 2, 3$ i.e e_i takes the values $0, 1, 2, 3$.

The co efficient of x^{14} can be formed with $e_i = 0, 1, 2, 3$.

i) when we take five 0's, one 2, four 3's i.e $\frac{10!}{5!\,1!\,4!}$

(ii) when we take four 0's one 1s, two 2's three 3's i.e $\frac{10!}{4!\,1!\,2!\,3!}$

(iii) when we take three 0's and seven 2's i.e $\frac{10!}{3!\,7!}$

Hence the co efficient of x^{14} in $(1+x+x^2+x^3)^{10}$ is

$$= \frac{10!}{5!\,1!\,4!} + \frac{10!}{4!\,1!\,2!\,3!} + \frac{10!}{3!\,7!}$$

Find a generating function for a_r = The number of non negative integral solutions of $e_1+e_2+e_3+e_4+e_5 = r$ where $0 \le e_1 \le 3$, $0 \le e_2 \le 3$, $2 \le e_3 \le 6$, $2 \le e_4 \le 6$, e_5 is odd and $1 \le e_5 \le 9$.

Sol:- Given that $e_1+e_2+e_3+e_4+e_5 = r$.

The given constraints are $0 \le e_1 \le 3$, $0 \le e_2 \le 3$, $2 \le e_3 \le 6$, $2 \le e_4 \le 6$,

e_5 is odd and $1 \le e_5 \le 9$.

e_1, e_2 can takes the values $0, 1, 2, 3$,

e_3, e_4 can takes the values, $2, 3, 4, 5, 6$.

e_5 is odd and $1 \le e_5 \le 9$, e_5 can take the values $1, 3, 5, 7, 9$.

Let us take
$$f_1(x) = x^0 + x^1 + x^2 + x^3 = 1 + x + x^2 + x^3$$
$$f_2(x) = x^0 + x^1 + x^2 + x^3 = 1 + x + x^2 + x^3$$
$$f_3(x) = x^2 + x^3 + x^4 + x^5 + x^6$$
$$f_4(x) = x^2 + x^3 + x^4 + x^5 + x^6$$
$$f_5(x) = x^1 + x^3 + x^5 + x^7 + x^9.$$

The generating function for the sequence a_r is given by.
$$f(x) = f_1(x)\, f_2(x)\, f_3(x)\, f_4(x)\, f_5(x)$$
$$= (1+x+x^2+x^3)(1+x+x^2+x^3)(x^2+x^3+x^4+x^5+x^6)(x^2+x^3+x^4+x^5+x^6)(x+x^3+x^5+x^7+x^9)$$
$$= (1+x+x^2+x^3)^2\, x^5 (1+x+x^2+x^3+x^4)^2 (1+x^2+x^4+x^6+x^8)$$
$$= x^5 (1+x+x^2+x^3)^2 (1+x+x^2+x^3+x^4)^2 (1+x+x^4+x^6+x^8) \quad \text{—(1)}$$

We know that $1 + x + x^2 + \dots + x^n = \dfrac{1-x^{n+1}}{1-x}$.

$$1 + x + x^2 + x^3 = \frac{1-x^4}{1-x} \quad \text{—(2)}$$

$$1 + x + x^2 + x^3 + x^4 = \frac{1-x^5}{1-x} \quad \text{—(3)}$$

sub. (2), (3) in (1), we get

$$f(x) = x^5 \left(\frac{1-x^4}{1-x}\right)^2 \left(\frac{1-x^5}{1-x}\right)^2 (1+x+x^4+x^6+x^8)$$

$$\therefore f(x) = \frac{x^5 (1-x^4)^2 (1-x^5)^2}{(1-x)^4} (1+x^2+x^4+x^6+x^8).$$

Find a generating function for a_r = the number of ways of distributing r similar balls into n- numbered boxes where each box is non empty.

Sol: The generating function for a_r = the number of ways of distributing r similar balls into n- numbered boxes.

Let an integral solution of an equation by counting the number of integral solutions to $e_1+e_2+e_3+\dots+e_n = r$ where each $e_i \geq 1$.

$\because$ Each box is nonempty

Here $e_1\, e_2\, e_3 \dots e_n$ are takes the values $1, 2, 3, \dots$

Let $f_1(x) = x + x^2 + x^3 + \dots$

$f_2(x) = x + x^2 + x^3 + \dots$

$\dots$

$f_n(x) = x + x^2 + x^3 + \dots$

The generating function for the sequence a_r is

$$f(x) = f_1(x)\, f_2(x)\, f_3(x) \dots f_n(x)$$

$$= (x+x^2+x^3+\dots)(x+x^2+x^3+\dots)\dots(x+x^2+x^3+\dots)$$

$$= (x+x^2+x^3+\dots)^n$$

$$= x^n (1+x+x^2+\dots)^n$$

$$= x^n ((1-x)^{-1})^n$$

$$f(x) = x^n (1-x)^{-n}$$

which is the required generating function.

Write the generating function for $a_r = \frac{(-1)^r (r+2)(r+1)}{r!}$ (42)

Sol: Given that $a_r = \frac{(-1)^r (r+2)(r+1)}{r!}$

The generating function $f(x)$ for the sequence a_r is given by

$$f(x) = \sum_{r=0}^{\infty} a_r \cdot x^r$$

$$f(x) = \sum_{r=0}^{\infty} \frac{(-1)^r (r+2)(r+1)}{r!} x^r \quad \text{--- (1)}$$

$$a_r = \frac{(-1)^r (r+2)(r+1)}{r!}$$

$r=0$, $a_0 = \frac{(-1)^0 2.1}{0!}$ $\qquad$ $r=1$, $a_1 = \frac{(-1)^1 (1+2)(1+1)}{1!} = -\frac{3.2}{1}$

$r=2$, $a_2 = \frac{(-1)^2 (2+2)(2+1)}{2!} = \frac{4.3}{2!}$ $\qquad$ $r=3$, $a_3 = \frac{(-1)^3 (3+2)(3+1)}{3!} = -\frac{5.4}{3!}$

$r=4$, $a_4 = \frac{(-1)^4 (4+2)(4+1)}{4!} = \frac{6.5}{4!}$ $\qquad$ $r=5$, $a_5 = \frac{(-1)^5 (5+2)(5+1)}{5!} = -\frac{7.6}{5!}$

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Sub. all these values in (1), we get

$$f(x) = 2 - \frac{3.2}{1} x + \frac{4.3}{2!} x^2 + \frac{5.4}{3!} x^3 + \frac{6.5}{4!} x^4 - \frac{7.6}{5!} x^5 + \cdots$$

$$= 2\left[1 - 3x + \frac{3.2}{1.2} x^2 + \frac{5.2}{3!} x^3 + \frac{5.3}{4!} x^4 - \frac{7.3}{5!} x^5 + \cdots\right].$$

Write a generating function for a_r where a_r is the number of integers between 0 and 999 whose sum of digits is r.

Sol:- Given that a_r is the number of integers b/n 0 and 999.

Let x_1, x_2, x_3 are positions of the digits

$$x_1 + x_2 + x_3 = r.$$

Here $0 \leq x_1 \leq 9, 0 \leq x_2 \leq 9, 0 \leq x_3 \leq 9$.

Let us take $f_1(x) = x^0 + x^1 + x^2 + \dots + x^9$

$f_2(x) = x^0 + x^1 + x^2 + \dots + x^9$

$f_3(x) = x^0 + x^1 + x^2 + \dots + x^9$

∴ The generating function is

$$f(x) = f_1(x) f_2(x) f_3(x)$$

$$= (1+x+x^2+\dots+x^9)(1+x+x^2+\dots+x^9)(1+x+x^2+\dots+x^9)$$

$$f(x) = (1+x+x^2+\dots+x^9)^3.$$

We know that $1+x+x^2+\dots+x^n = \frac{1-x^{n+1}}{1-x}$.

$$= \left(\frac{1-x^{10}}{1-x}\right)^3$$

$$= \frac{(1-x^{10})^3}{(1-x)^3}$$

$$f(x) = (1-x^{10})^3 (1-x)^{-3}.$$

Which is the required generating function for a_r.

Write a generating function for a_r when a_r is (43)

(a) the number of ways of selecting r balls from 3 red balls, 5 blue balls, 7 white balls.

(b) the number of integers between 0 and 999 whose sum of digits is r.

sol:- (a). Let x_1 be the number of red balls selecting, x_2 be the no. of blue balls selecting and x_3 be the no. of white balls selecting.

Then $x_1 + x_2 + x_3 = r$.

Here $0 \le x_1 \le 3$, $0 \le x_2 \le 5$, and $0 \le x_3 \le 7$.

Let us take $f_1(x) = x^0 + x^1 + x^2 + x^3$

$f_2(x) = x^0 + x + x^2 + \cdots + x^5$

$f_3(x) = x^0 + x + \cdots + x^7$.

$\therefore$ The generating function is.

$$f(x) = f_1(x) f_2(x) f_3(x)$$

$$f(x) = (1+x+x^2+x^3)(1+x+x^2+\cdots+x^5)(1+x+\cdots+x^7)$$

(b). Let x_1, x_2, x_3 are digits. Then $x_1 + x_2 + x_3 = r$.

Here $0 \le x_1 \le 9$ $0 \le x_2 \le 9$ $0 \le x_3 \le 9$.

Let us take $f_1(x) = x^0 + x^1 + \cdots + x^9$.

$f_2(x) = x^0 + x^1 + \cdots + x^9$

$f_3(x) = x^0 + x^1 + \cdots + x^9$

The generating function is

$$f(x) = f_1(x) f_2(x) f_3(x)$$

$$f(x) = (1+x+x^2+\cdots x^9)^3.$$

The Pigeonhole Principle :—

If m pigeons occupy n pigeonholes and if $m>n$ then two or more pigeons occupy the same pigeon hole.

This is often restated as follows:

If m pigeons occupy n pigeonholes where $m>n$ then at least one pigeonhole must contain two or more pigeons in it. (OR).

Let $f: X \to Y$ where X and Y are finite sets $|X|=m$ $|Y|=n$ and $m>n$.

Then there exists atleast two distinct elements x_1 and x_2 in X such that $f(x_1)=f(x_2)$.

Proof :- Let $X=\{x_1\ x_2\ x_3 \ldots x_m\}$ suppose f is injective Then $f(x_1)\ f(x_2) \ldots f(x_m)$ are distinct elements in Y. so $m \leq n$.

But this contradicts the assumption that $m>n$.

$\therefore$ f is not injective and there must be atleast two distinct elements x_1 and x_2 such that $f(x_1)=f(x_2)$.

→ The pigeonhole principle is also called the Dirichlet Box Principle.

~~other~~ Another form of the pigeonhole principle :—

If A is the average number of pigeons per hole, then some pigeonhole contains at least A pigeons and some pigeonhole contains at most A pigeons.

If n is the number of pigeonholes and m_i is the number of pigeons in the i^{th} hole, then we prove that either $m_1 \geq A$ or $m_2 \geq A$ or ... or $m_n \geq A$

Let us assume the contrary, namely that $m_1<A$ and $m_2<A \ldots$ and $m_n<A$.

But then the sum $m_1+m_2+m_3+\cdots+m_n < nA$ = the total number of pigeons.

This clearly contradiction since $m_1+m_2+m_3+\cdots+m_n$ also equals the the total number of pigeons.

→ The number of pigeons in a pigeonhole is necessarily an integer, but the average A need not be an integer.

If A is the average number of pigeons per hole, then some pigeonhole contains at least $\lceil A \rceil$ pigeons and some pigeonhole contains at most $\lfloor A \rfloor$ pigeons.

Generalization of Pigeonhole principle :-

If m pigeons occupy n pigeonholes then at least one pigeonhole must contain $(p+1)$ or more pigeons where $p = \lfloor (m-1)/n \rfloor$

Proof :- We prove this principle by the method of contradiction.

Assume that the conclusion part of the principle is not true.

Then, no pigeonhole contains $(p+1)$ or more pigeons.

This means that every pigeonhole contains p or less number of pigeons.

$$\text{Total number of pigeons} \le np = n\lfloor (m-1)/n \rfloor \le n\left(\frac{m-1}{n}\right) = (m-1)$$

This is a contradiction, because the total number of pigeons is m.

Hence our assumption is wrong and the principle is true.

Applications of the pigeonhole principle :—

1. If $n+1$ pigeons are distributed among n pigeonholes then some hole contains atleast 2 pigeons. If $2n+1$ pigeons are distributed among n pigeonholes then some pigeon hole contains at least 3 pigeons. In general, if k is an integer and $kn+1$ pigeons are distributed among ~~then~~ so n pigeonholes, then some hole contains at least $k+1$ pigeons. This follows since the average number of pigeons per hole is $k+1/n$ and $\lceil k+1/n \rceil = k+1$.

2. In any group of 367 people there must be at least one pair with the same birthday.

3. If 4 different pairs of socks are scrambled in a drawer, one need only select 5 individual socks in order to guarantee finding a matching pair. Here the pairs determine 4 pigeonholes and 5 individual socks in 4 holes implies a matching pair.

4. In a group of 61 people at least 6 people were born in the same month.

5. If 401 letter were delivered to 50 appartments then some apartment received atmost 8 letters.

6. Suppose 50 chairs are arranged in a rectangular array of 5 rows and 10 columns. suppose that 41 students are seated randomly in the chairs (one student per chair) Then some row contains at least 9 students. Some column contains at least 5 students, some row contains atmost 8 students and some column contains at most 4 students. The result follows from the pigeonhole principle because the average number of students per row is 8.2 and the average number per ~~number~~ column is 4.1

(7) If $x_1, x_2, x_3 \dots x_8$ are 8 distinct integers then there is some pair of these integers with the same remainder when divided by 7.

If each integer is divided by 7 and their remainders are recorded, the [illegible] for these remainders are 0, 1, 2, 3, 4, 5 and 6. Thus we have 8 remainders are to be distributed among the [illegible] possibilities from [illegible]

In otherwords. it is like distributing 8 pigeons among 7 pigeon holes.

We conclude that there are at least 2 pigeons in some hole or there are at least 2 of the remainders which are equal.

Applying pigeon hole principle show that of any 14 integers are 49
Selected from the set $S = \{1, 2, 3, \ldots 25\}$ there are at least two whose
sum is 26. Also write a statement that generalizes this result.

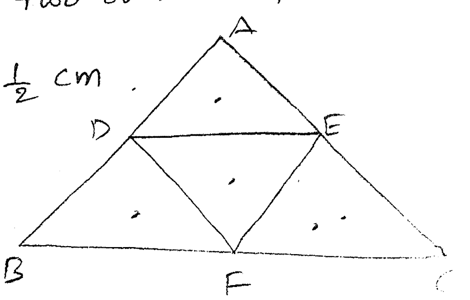
Sol: The different sets, each containing two numbers whose sum is equal to 26 are $\{1, 25\}$ $\{2, 24\}$ ... $\{12, 14\}$ $\{13\}$. These 13 sets can be thought of as pigeonhole and 14 chosen number as pigeon. Since $13 < 14$ i.e the number of pigeonholes less than the number of pigeons, by pigeon hole principle we can conclude that two of the selected numbers must belong to the same set whose sum is 26.

(b) If $S = \{1, 2, 3, \ldots 2n+1\}$ for a +ve integer n. Then any subset of size $n+2$ from S must contain at least one two element subset whose sum is $2n+2$.

ABC is an equilateral triangle whose sides are of length 1 cm each. If we select 5 points inside the triangle prove that at least two of these points are such that the distance between them is less than $\frac{1}{2}$ cm.

Sol: Consider the triangle DEF formed by the mid points of the sides BC, CA and AB of the given triangle ABC Then the triangle ABC is partitioned into four small equilateral triangles, each of which has sides equal to $\frac{1}{2}$ cm. Treating each of these four portions as pigeonhole and five points choosen inside the triangle as pigeons, we find by using the pigeon hole principle that at least one portion must contain two or more points.

$\therefore$ The distance b/n such points is less than $\frac{1}{2}$ cm



Prove that in any set of 29 persons at least 5 persons must have been born on the same day of the week.

Sol: Treating the seven days of a week as 7 pigeonholes and 29 persons as pigeons. We find by using the generalized pigeonhole principle that at least one day of the week is assigned to $\left\lfloor \frac{29-1}{7} \right\rfloor + 1 = 5$ or more. $\frac{m-1}{n}+1$ persons. In other words, atleast 5 of any 29 persons must have been born on the same day of the week.

How many persons must be choosen in order that at least five of them will have birth days in the same calendar month?

Sol: Let n be the required number of persons. since the number of months over which the birthdays are distributed is 12, the least number of persons who have their birthdays in the same month is, by the generalized pigeonhole principle equal to $\left\lfloor \frac{n-1}{12} \right\rfloor + 1$. This number is 5.

if $\left\lfloor \frac{n-1}{12} \right\rfloor + 1 = 5$ or $n = 49$.

$\therefore$ The number of persons is 49 (at least)

If we select any group of 1000 students on campus show that atleast three of them must have the same birthday.

Sol:- The maximum number of days in a year is 366.

Think of students as pigeons and days of the year as pigeon holes.

Then, by the generalized pigeonhole principle, the maximum number of students having the same birth day is $\left\lfloor \frac{1000-1}{366} \right\rfloor + 1 = 2 + 1 = 3$.

Solve $a_n = 3a_{n-1} + 2a_{n-2} + (n+3)\,3^n$.

Sol:- Given that $a_n - 3a_{n-1} - 2a_{n-2} = (3+n)\,3^n$. — ①.

The homogeneous part of the given recurrence relation is

$$a_n - 3a_{n-1} + 2a_{n-2} = 0 \quad \text{— ②}$$

Let $a_n = ck^n$ — ③ where $c \neq 0$ $k \neq 0$ be the solution of relation ②

sub ③ in ②, we get

$$ck^n - 3ck^{n-1} - 2ck^{n-2} = 0.$$

$$ck^n\left[1 - \frac{3}{k} - \frac{2}{k^2}\right] = 0.$$

$$\frac{ck^n\left[k^2 - 3k - 2\right]}{k^2} = 0.$$

$$ck^{n-2} \neq 0 \quad k^2 - 3k - 2 = 0.$$

Which is the characteristic equation of the relation ②.

$$k = \frac{3 \pm \sqrt{9+8}}{2} = \frac{3 \pm \sqrt{17}}{2}$$

$$k_1 = \frac{3+\sqrt{17}}{3} \qquad k_2 = \frac{3-\sqrt{17}}{2}.$$ The roots are real and distinct.

The general solution of the relation ② is

$$a_n^{(h)} = A_0 k_1^n + A_1 k_2^n.$$

$$a_n^{(h)} = A_0\left(\frac{3+\sqrt{17}}{3}\right)^n + A_1\left(\frac{3-\sqrt{17}}{2}\right)^n.$$

We observe that the R.H.S of equation is of the form $f(n) = (n+3)\,3^n$

i.e $f(n) = \phi(n).\,b^n$.

Since 3 is not characteristic root of the associated homogeneous relation.

Let $a_n^{(p)} = (A_0 + A_1 n)\,3^n$.

Substitute all these values in (1), we get.

$$n(A_0+A_1 n)3^n - \left[(n-1)\left[A_0+A_1(n-1)\right]3^{n-1}\right] - 6(n-2)\left[A_0+A_1(n-2)\right]3^{n-2} = (n+1)3^n.$$

$$\left[A_0 n + A_1 n^2\right]3^n - \left[A_0(n-1)+A_1(n-1)^2\right]3^{n-1} - \left[6A_0(n-2)+6A_1(n-2)^2\right]3^{n-2} = (n+1)3^n$$

$$\left[A_0 n + A_1 n^2\right]3^n - \left[A_0(n-1)+A_1(n^2+1-2n)\right]\frac{3^n}{3} - \frac{3^n}{9}\left[6A_0 n - 12A_0 + 6A_1(n^2+4-4n)\right]. = n3^n + 3^n.$$

$$\left[A_0 n - A_1 n^2\right] - \frac{\left[A_0 n + A_0 + A_1 n^2 + A_1 + 2A_1 n\right]}{3} - \frac{\left[6A_0 n - 12A_0 + 6A_1 n^2 + 24A_1 - 24A_1 n\right]}{9} = (n+1).$$

$$9A_0 n + 9A_1 n^2 - 3A_0 n + 3A_0 - 3A_1 n^2 - 3A_1 + 6A_1 n - 6A_0 n + 12A_0 - 6A_1 n^2.$$
$$-24A_1 + 24A_1 n = 9n+9.$$

$$-18A_1 n^2 + 30A_1 n + 15A_0 - 27A_1 = 9n+9.$$

Comparing the co efficient of n and constant terms,

we get $30A_1 = 9 \Rightarrow A_1 = \frac{9}{30} = \frac{3}{10}.$

$15A_0 - 27A_1 = 9$

$15A_0 - 27.\frac{3}{10} = 9. \Rightarrow 15A_0 = 9 + \frac{81}{10}.$

$15A_0 = \frac{171}{10}$

$A_0 = \frac{\cancel{171}\ 57}{10.\cancel{15}\ 5}$

$A_0 = \frac{57}{50}.$

$\therefore a_n^{(p)} = n\left(\frac{57}{50} + \frac{3}{10}n\right)3^n$

$\therefore$ The general solution of (1) is $a_n = a_n^{(h)} + a_n^{(p)}$.

$$a_n = A_0(-2)^n + A_1 3^n + n\left(\frac{57}{50} + \frac{3}{10}n\right)3^n.$$

Find the co efficient of x^{23} and x^{32} in $(1+x^5+x^9)^{10}$.

sol:- Given that $(1+x^5+x^9)^{10}$

(i) To find the co efficient of x^{23}

i.e $e_1+e_2+e_3+e_4+\dots+e_{10}=23$.

$e_i = 0, 5, 9$. $\because$ e_i takes the values 0, 5, 9.

The co efficient of x^{23} can be formed with $e_i = 0, 5, 9$ when we take two 9's and one 5 and remaining seven 0's.

$\therefore$ The co efficient of x^{23} in $(1+x^5+x^9)^{10}$ is $\dfrac{10!}{2!\,1!\,7!}$.

(ii) To find the co efficient of x^{32}.

i.e $e_1+e_2+e_3+\dots+e_{10}=32$.

$e_i = 0, 5, 9$ e_i takes the values 0, 5, 9.

The co efficient of x^{32} can be formed with $e_i=0, 5, 9$. when we take three 9's, one's and remaining six 0's.

$\therefore$ The co efficient of x^{32} in $(1+x^5+x^9)^{32}$ is. $\dfrac{10!}{3!\,1!\,6!}$

Find the co efficient of x^{16} in $(1+x^4+x^8)^{10}$.

sol: Given that $(1+x^4+x^8)^{10}$.

To find the co efficient of x^{16}.

i.e $e_1+e_2+e_3+\dots+e_{10}=16$.

$e_i = 0, 4, 8$ e_i takes the values 0, 4, 8.

The co efficient of x^{16} can be formed with $e_i=0, 4, 8$ when we take four 4's, no 8's and six 0's; two 8's, no 4's and eight 0's; two 4's, one 8 and seven 0's.

$\therefore$ The co efficient of x^{16} in $(1+x^4+x^8)^{10}$ is $\dfrac{10!}{4!\,6!}+\dfrac{10!}{2!\,8!}+\dfrac{10!}{2!\,1!\,7!}$

Find the co efficient of x^9 and x^{25} in $(1+x^3+x^8)^{10}$.

Sol:- Given that $(1+x^3+x^8)^{10}$.

(i) To find the co efficient of x^9.

i.e $e_1+e_2+e_3+\dots+e_{10}=9$.

$e_i = 0, 3, 8$ e_i takes the values 0, 3, 8.

The co efficient of x^9 can be formed with $e_i = 0, 5, 9$ when we take three 3's and remaining seven 0's.

$\therefore$ The co efficient of x^9 in $(1+x^3+x^8)^{10}$ is $\dfrac{10!}{3!\,7!}$.

(ii) To find the co efficient of x^{25}.

i.e $e_1+e_2+e_3+\dots+e_{10}=25$.

$e_i = 0, 3, 8$ e_i takes the values 0, 3, 8.

The co efficient of x^{25} can be formed with $e_i = 0, 5, 9$ when we take three 3's, two 8's and remaining five 0's.

$\therefore$ The co efficient of x^{25} in $(1+x^3+x^8)^{10}$ is $\dfrac{10!}{3!\,2!\,5!}$

In $(1+x^5+x^9)^{10}$ find (a) the co efficient of x^{23} (b) the co efficient of x^{32} (40)

Sol:- Given that $(1+x^5+x^9)^{10}$.

(i) To find the co efficient of x^{23}.

i.e $e_1+e_2+e_3+\cdots+e_{10}=23$.

$e_i=0,5,9$.

The co efficient of x^{23} can be formed with $e_i=0,5,9$.

When we take two 9's, one 5 and seven 0's.

Hence the co efficient of x^{23} in $(1+x^5+x^9)^{10}$ is $\dfrac{10!}{2!\,1!\,7!}$

(ii) To find the co efficient of x^{32}.

i.e $e_1+e_2+e_3+\cdots+e_{10}=32$.

$e_i=0,5,9$.

The co efficient of x^{32} can be formed with $e_i=0,5,9$.

When we take three 9's, one 5 and six 0's

Hence the co efficient of x^{32} in $(1+x^5+x^9)^{10}$ is $\dfrac{10!}{3!\,1!\,6!}$.

→ Determine the co efficient of x^5 in $(a+bx+cx^2)^{10}$

Sol:- Given that $(a+bx+cx^2)^{10}$.

To find the co efficient of x^5.

i.e $e_1+e_2+e_3+\cdots+e_{10}=5$

$e_i=0,1,2$ $\quad$ e_i takes the values 0, 1, 2.

The co efficient of x^5 can be formed with $e_i=0,1,2$.

(i) When we take two 2's, one 1 and seven 0's i.e $\dfrac{10!}{2!\,1!\,7!}a^7c^2b$.

(ii) When we take three 1's, one 2 and six 0's i.e $\dfrac{10!}{3!\,1!\,6!}b^3ca^6$.

(iii) When we take five 0's, five 1's i.e $\dfrac{10!}{5!\,5!}a^5b^5$.

Hence the co efficient of x^5 in $(a+bx+cx^2)^{10}$ is

$$\frac{10!}{7!\,1!\,2!}a^7bc^2+\frac{10!}{6!\,3!\,1!}a^6b^3c+\frac{10!}{5!\,5!}a^5b^5$$

Find the co efficient of x^9 and x^{25} in $(1+x^3+x^8)^{10}$.

Sol:- Given that $(1+x^3+x^8)^{10}$.

(i) To find the co efficient of x^9.

i.e $e_1+e_2+e_3+\dots+e_{10}=9$.

$e_i = 0, 3, 8$ $\quad$ e_i takes the values 0, 3, 8.

The co efficient of x^9 can be formed with $e_i = 0, 5, 9$ when we take three 3's and remaining seven 0's.

$\therefore$ The co efficient of x^9 in $(1+x^3+x^8)^{10}$ is $\dfrac{10!}{3!\,7!}$.

(ii) To find the co efficient of x^{25}.

i.e $e_1+e_2+e_3+\dots+e_{10}=25$.

$e_i = 0, 3, 8$ $\quad$ e_i takes the values 0, 3, 8.

The co efficient of x^{25} can be formed with $e_i = 0, 5, 9$ when we take. three 3's, two 8's and remaining five 0's.

$\therefore$ The co efficient of x^{25} in $(1+x^3+x^8)^{10}$ is $\dfrac{10!}{3!\,2!\,5!}$

Find the co efficient of $x^3y^2z^3$ in the expansion of $(x+y+z)^8$.

sol:- We know that For any positive integers n and t,

The co efficient of $x_1^{n_1}x_2^{n_2}x_3^{n_3}\cdots x_t^{n_t}$ in the expansion of $(x_1+x_2+\cdots+x_t)^n$ is

$\dfrac{n!}{n_1!\,n_2!\,n_3!\cdots n_t!}$ Where each n_i is a non-ve integer $\le n$ and

$n_1+n_2+n_3+\cdots+n_t=n$.

$\therefore$ The co-efficient of $x^3y^2z^3$ in $(x+y+z)^8$ is $=\dfrac{8!}{3!\,2!\,3!}$

Find the co efficient of $x^5y^{10}z^5w^5$ in $(x+7y+3z+w)^{25}$.

sol:- We know that For any +ve integer n and t,

The co efficient of $x_1^{n_1}x_2^{n_2}x_3^{n_3}\cdots x_t^{n_t}$ in the expansion of $(x_1+x_2+\cdots+x_t)^n$

is $\dfrac{n!}{n_1!\,n_2!\,n_3!\cdots n_t!}$ Where n_i is a non-ve integer $\le n$ and

$n_1+n_2+n_3+\cdots+n_t=n$.

$\therefore$ The co efficient of $x^5y^{10}z^5w^5$ in $(x+7y+3z+w)^{25}$ is

$$\frac{25!}{5!\,10!\,5!\,5!}\,7^{10}.\,3^5.$$

Find the term which contains x^{11} and y^4 in the expansion of $(2x^3-3xy^2+z^2)^6$

sol:- Given that $(2x^3-3xy^2+z^2)^6$.

The co efficient of $x_1^{n_1}x_2^{n_2}x_3^{n_3}\cdots x_t^{n_t}$ in the expansion of $(x_1+x_2+\cdots+x_t)^n$

is $\dfrac{n!}{n_1!\,n_2!\,n_3!\cdots n_t!}$ Where each n_i is a non-ve integer $\le n$ and

$n_1+n_2+n_3+\cdots+n_t=n$.

sol:- By the multinomial theorem, the general term in the given expansion i

$$\binom{6}{n_1\ n_2\ n_3}(2x^3)^{n_1}(-3xy^2)^{n_2}(z^2)^{n_3}=\binom{6}{n_1\ n_2\ n_3}2^{n_1}(-3)^{n_2}x^{3n_1+n_2}y^{2n_2}z^{2n_3}.$$

Thus, for the term containing x^{11} and y^4 we should have $3n_1+n_2=11$

and $2n_2=4$ so that $n_1=3$ and $n_2=2$ since $n_1+n_2+n_3=6$ we should

then have $n_3=1$.

The term containing x^{11} and y^4 is

$$\binom{6}{3,\ 2,\ 1} 2^3 (-3)^2 x^{11} y^4 z^2 = \left\{ \frac{6!}{3!\,2!\,1!} 8 \times 9 \right\} x^{11} y^4 z^2 = 4320\, x^{11} y^4 z^2$$

$\therefore$ The generating function is $f(x) = f_1(x) f_2(x) f_3(x) \cdots f_6(x)$.

$$f(x) = (x+x^2+\cdots+x^5)(x^2+x^3+\cdots)^5.$$

$$f(x) = x(1+x^2+\cdots+x^4)x^{10}(1+x+x^2+\cdots)^5.$$

$$f(x) = x^{11}(1+x^2+\cdots+x^4)[(1-x)^{-1}]^5$$

$$= x^{11}(1+x+x^2+x^3+x^4)(1-x)^{-5}$$

We know that $1+x+x^2+\cdots+x^n = \dfrac{1-x^{n+1}}{1-x}.$

$$= x^{11}\cdot\left(\frac{1-x^5}{1-x}\right)(1-x)^{-5}$$

$$= x^{11}(1-x^5)(1-x)^{-6}.$$

We know that If n is a positive integer, $(1-x)^{-n} = \sum_{r=0}^{\infty}\binom{n+r-1}{r}x^r$

$$= x^{11}(1-x^5)\sum_{r=0}^{\infty}\binom{6+r-1}{r}x^r$$

$$= (x^{11}-x^{16})\sum_{r=0}^{\infty}\binom{5+r}{r}x^r$$

$$= \sum_{r=0}^{\infty}\binom{5+r}{r}x^{r+11} - \sum_{r=0}^{\infty}\binom{5+r}{r}x^{16+r}.$$

$\therefore$ The co efficient of x^{20} in this last product is

$$-\binom{14}{9}-\binom{9}{4}.$$

Find the co efficient of x^{20} in $x^4(1+x+x^2+x^3)(1+x+\cdots+x^4).(1+x+x^2+\cdots+x^{12})$.

Sol: $x^4(1+x+x^2+x^3)(1+x+x^2+x^3+x^4)(1+x+x^2+\cdots+x^{12}) =$

We know that $1+x+x^2+\cdots+x^n = \dfrac{1-x^{n+1}}{1-x}.$

$$= x^4\left(\frac{1-x^4}{1-x}\right)\left(\frac{1-x^5}{1-x}\right)\left(\frac{1-x^{13}}{1-x}\right)$$

$$= x^4(1-x^4)(1-x^5)(1-x^{13})(1-x)^{-3}.$$

Find the co efficient of x^{20} in $(x^3+x^4+x^5+\dots)^5$

Sol:- Given that $(x^3+x^4+x^5+\dots)^5 = [x^3(1+x+x^2+\dots)]^5$

$$= x^{15}(1+x+x^2+\dots)^5$$

$$= x^{15}[(1-x)^{-1}]^5 = x^{15}(1-x)^{-5}$$

$$[\because \text{WKT } (1-x)^{-1} = 1+x+x^2+x^3+\dots]$$

If n is a positive Integer, Then $(1-x)^{-n} = \sum_{r=0}^{\infty}\binom{n+r-1}{r}x^r$.

$$(x^3+x^4+x^5+\dots)^5 = x^{15}\sum_{r=0}^{\infty}\binom{5+r-1}{r}x^r$$

$$= x^{15}\sum_{r=0}^{\infty}\binom{4+r}{r}x^r$$

$$= \sum_{r=0}^{\infty}\binom{4+r}{r}x^{15+r}.$$

The co eff. of x^{20} in $(x^3+x^4+x^5+\dots)^5$ is $\binom{9}{5} = 9_{c_5}$ $[\because r=5]$

$$9_{c_5} = \frac{9!}{4!\,5!} = \frac{9\times8\times7\times6\times5!}{4!\times5!}$$

$$=$$

Find the number of ways of placing 20 similar balls into 6 numbered boxes. so that the first box contains any no. of balls between 1 and 5 inclusive and the other 5 boxes must contain 2 or more balls each.

Sol:- Given that the total no. of balls 20 and no. of boxes 6.

Let x_1 be the first box contains any no. of balls b/n 1 and 5 inclusive.

and let x_2, x_3, x_4, x_5, x_6 are other boxes contain 2 or more balls.

We count the number of integral solutions to $x_1+x_2+x_3+x_4+x_5+x_6=20$

Where $1 \le x_1 \le 5$ and $2 \le x_2, x_3, \dots x_6$.

Let $f_1(x) = x+x^2+x^3+x^4+x^5$

$f_2(x) = x^2+x^3+x^4+\dots$

$\dots$

$f_6(x) = x^2+x^3+x^4+\dots$

We know that If n is a positive integer $(1-x)^{-n} = \sum_{r=0}^{\infty} \binom{n+r-1}{r} x^r$ (6)

$$= x^4 (1-x^4)(1-x^5)(1-x^{13}) \sum_{r=0}^{\infty} \binom{3+r-1}{r} x^r$$

$$= x^4 (1-x^5-x^4-x^{11})(1-x^{13}) \sum_{r=0}^{\infty} \binom{2+r}{r} x^r .$$

$$= x^4 (1-x^{13}-x^5-x^{18}-x^4-x^{17}-x^{11}-x^{24}) \sum_{r=0}^{\infty} \binom{2+r}{r} x^r$$

$$= (x^4 - x^8 - x^9) \sum_{r=0}^{\infty} \binom{2+r}{r} x^r$$ (Neglecting remaining terms because those powers of x. greater than 10)

$$= \sum_{r=0}^{\infty} \binom{2+r}{r} x^{r+4} - \sum_{r=0}^{\infty} \binom{2+r}{r} x^{r+8} - \sum_{r=0}^{\infty} \binom{2+r}{r} x^{r+9}$$

The co efficient of x^{10} in last product is .

$$= \binom{8}{6} - \binom{4}{2} - \binom{3}{1}.$$

Find the co efficient of x^{12} in $\frac{(1-x^4-x^7+x^{11})}{(1-x)^5}$.

Sol: Given that $\frac{1-x^4-x^7+x^{11}}{(1-x)^5} = (1-x^4-x^7+x^{11})(1-x)^{-5}$

Wkt If n is a positive integer, $(1-x)^{-n} = \sum_{r=0}^{\infty} \binom{n+r-1}{r} x^r$.

$$(1-x^4-x^7+x^{11})(1-x)^{-5} = (1-x^4-x^7+x^{11}) \sum_{r=0}^{\infty} \binom{5+r-1}{r} x^r$$

$$= \sum_{r=0}^{\infty} \binom{4+r}{r} x^r - \sum_{r=0}^{\infty} \binom{4+r}{r} x^{r+4} - \sum_{r=0}^{\infty} \binom{4+r}{r} x^{r+7} + \sum_{r=0}^{\infty} \binom{4+r}{r} x^{r+11}$$

$\therefore$ The co efficient of x^{12} in this last product is .

$$= \binom{16}{12} - \binom{12}{8} - \binom{9}{5} + \binom{5}{1}$$

Find the co efficient of x^9 and x^{25} in $(1+x^3+x^8)^{10}$.

Sol:- Given that $(1+x^3+x^8)^{10}$.

(i) To find the co efficient of x^9.

i.e $e_1+e_2+e_3+\cdots+e_{10}=9$.

$e_i = 0, 3, 8$ $\quad$ e_i takes the values $0, 3, 8$.

The co efficient of x^9 can be formed with $e_i = 0, 5, 9$ when we take three 3's and remaining seven 0's.

$\therefore$ The co efficient of x^9 in $(1+x^3+x^8)^{10}$ is $\dfrac{10!}{3!\,7!}$.

(ii) To find the co efficient of x^{25}.

i.e $e_1+e_2+e_3+\cdots+e_{10}=25$.

$e_i = 0, 3, 8$ $\quad$ e_i takes the values $0, 3, 8$.

The co efficient of x^{25} can be formed with $e_i = 0, 5, 9$ when we take. three 3's, two 8's and remaining five 0's.

$\therefore$ The co efficient of x^{25} in $(1+x^3+x^8)^{10}$ is $\dfrac{10!}{3!\,2!\,5!}$

Find the co efficient of x^{23} and x^{32} in $(1+x^5+x^9)$.

sol: Given that $(1+x^5+x^9)^{10}$.

(i) To find the co efficient of x^{23}

i.e $e_1+e_2+e_3+e_4+\dots+e_{10}=23$.

$e_i = 0, 5, 9$. $\because$ e_i takes the values 0, 5, 9.

The co efficient of x^{23} can be formed with $e_i = 0, 5, 9$ when we take two 9's and one 5 and remaining seven 0's.

$\therefore$ The co efficient of x^{23} in $(1+x^5+x^9)^{10}$ is $\dfrac{10!}{2!\,1!\,7!}$.

(ii) To find the co efficient of x^{32}.

i.e $e_1+e_2+e_3+\dots+e_{10}=32$.

$e_1 = 0, 5, 9$ e_i takes the values 0, 5, 9.

The co efficient of x^{32} can be formed with $e_i=0, 5, 9$. when we take three 9's, one's 5 and remaining six 0's.

$\therefore$ The co efficient of x^{32} in $(1+x^5+x^9)^{32}$ is. $\dfrac{10!}{3!\,1!\,6!}$.

Find the co efficient of x^{16} in $(1+x^4+x^8)^{10}$.

sol: Given that $(1+x^4+x^8)^{10}$.

To find the co efficient of x^{16}.

i.e $e_1+e_2+e_3+\dots+e_{10}=16$.

$e_i = 0, 4, 8$ e_i takes the values 0, 4, 8.

The co efficient of x^{16} can be formed with $e_i = 0, 4, 8$ when we take four 4's, no 8's and six 0's ; two 8's, no 4's and eight 0's ; two 4's, one 8 and seven 0's.

$\therefore$ The co efficient of x^{16} in $(1+x^4+x^8)^{10}$ is $\dfrac{10!}{4!\,6!} + \dfrac{10!}{2!\,8!} + \dfrac{10!}{2!\,1!\,7}$

Find the co efficient of x^{14} in $(1+x+x^2+x^3)^{10}$.

Sol:- Given that $(1+x+x^2+x^3)^{10}$.

To find the co efficient of x^{14}

i.e $e_1+e_2+e_3+\cdots+e_{10} = 14$.

$e_i = 0, 1, 2, 3$ i.e e_i takes the values 0, 1, 2, 3.

The co efficient of x^{14} can be formed with $e_i = 0, 1, 2, 3$.

(i) When we take. five 0's, one 2, four 3's i.e $\frac{10!}{5!\,1!\,4!}$

(ii) When we take. four 0's one 1s, two 4's three 3's i.e $\frac{10!}{4!\,1!\,2!\,3!}$

(iii) When we take. three 0's and seven 2's i.e $\frac{10!}{3!\,7!}$

Hence the co efficient of x^{14} in $(1+x+x^2+x^3)^{10}$ is

$$= \frac{10!}{5!\,1!\,4!} + \frac{10!}{4!\,1!\,2!\,3!} + \frac{10!}{3!\,7!}$$

In $(1+x^5+x^9)^{10}$ find (a) the coefficient of x^{23} (b) the coefficient of x^{32} (40)

Sol:- Given that $(1+x^5+x^9)^{10}$.

(i) To find the coefficient of x^{23}.

i.e $e_1+e_2+e_3+\dots+e_{10}=23$.

$e_i=0,5,9$.

The coefficient of x^{23} can be formed with $e_i=0,5,9$.

When we take two 9's, one 5 and seven 0's.

Hence the coefficient of x^{23} in $(1+x^5+x^9)^{10}$ is $\frac{10!}{2!\,1!\,7!}$

(ii) To find the coefficient of x^{32}.

i.e $e_1+e_2+e_3+\dots+e_{10}=32$.

$e_i=0,5,9$.

The coefficient of x^{32} can be formed with $e_i=0,5,9$.

When we take three 9's, one 5 and six 0's

Hence the coefficient of x^{32} in $(1+x^5+x^9)^{10}$ is $\frac{10!}{3!\,1!\,6!}$.

→ Determine the coefficient of x^5 in $(a+bx+cx^2)^{10}$

Sol:- Given that $(a+bx+cx^2)^{10}$.

To find the coefficient of x^5.

i.e $e_1+e_2+e_3+\dots+e_{10}=5$.

$e_i=0,1,2$ e_i takes the values 0, 1, 2.

The coefficient of x^5 can be formed with $e_i=0,1,2$.

(i) When we take two 2's, one 1 and seven 0's i.e $\frac{10!}{2!\,1!\,7!}a^7c^2b$.

(ii) When we take three 1's, one 2 and six 0's i.e $\frac{10!}{3!\,1!\,6!}b^3ca^6$.

(iii) When we take five 0's, five 1's i.e $\frac{10!}{5!\,5!}a^5b^5$.

Hence the coefficient of x^5 in $(a+bx+cx^2)^{10}$ is

$$\frac{10!}{7!\,1!\,2!}a^7bc^2+\frac{10!}{6!\,3!\,1!}a^6b^3c+\frac{10!}{5!\,5!}a^5b^5$$